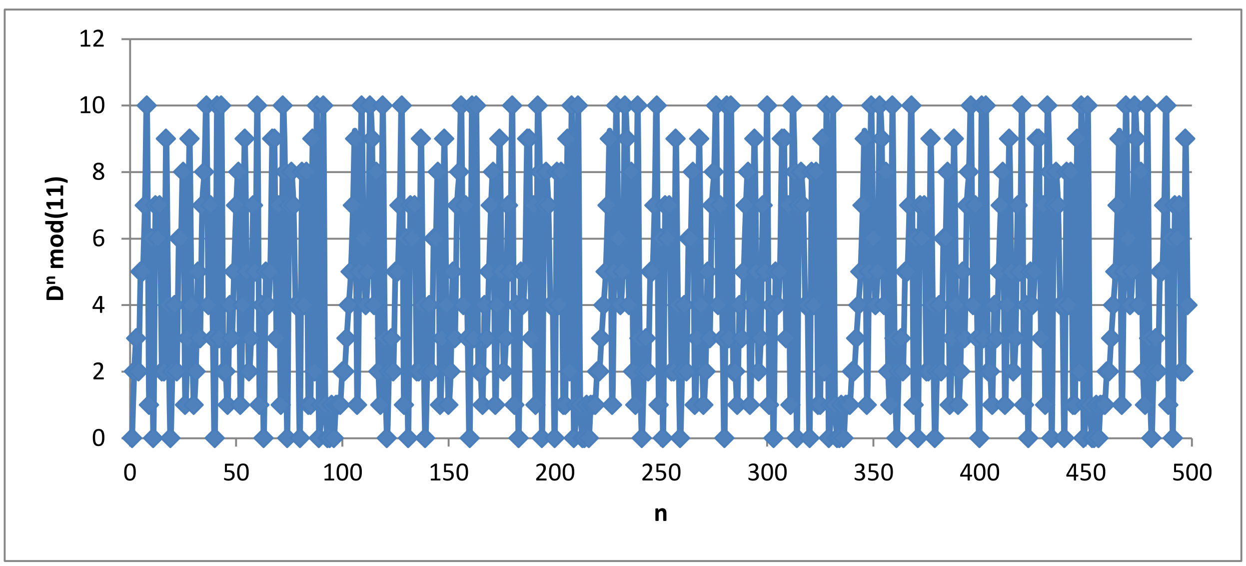
CHALKBOARD #4

MATRIX REPRESENTATIONS, CIRCULANT MATRICES and DETERMINANTS

Reference to ideas in this Chalkboard:
Irwin Kra and Santiago Simanca, **Notices of the AMS** Vol 59 (3) pp368-377 (2012)

Purpose: To give a brief introduction to the basic properties of matrices and matrix operations. These matrices will be used in future chalkboards to find periodic points and orbits under operations (automorphism) of the 3-torus.

Trace of Powers of the Diagonal Circulant Matrix D^n taken modulo 11 for integer values of n from 1 to 500. Note the Period of the Sequence P(11) = 120.



Define A as the 3 x3 Matrix of integers that shifts a number in the Perrin sequence to the next number. Also define the identity matrix I.

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next define the column vector to contain 3 x1 integers starting with the Perrin sequence

$$v := \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

3,0,2,3,2,5,...

By matrix operation A*v is a new vector:

$$A \cdot v = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Powers of A will shift the vector by m places:
e.g. m = 3

$$A^3 \cdot v = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$A^{24} \cdot v = \begin{pmatrix} 853 \\ 1130 \\ 1497 \end{pmatrix}$$

Note that
853 = 3 mod(5)
1130 = 0 mod(5)
1497 = 2 mod(5) so for
m= 24 and using numbers mod 5
the 24 th power defines the sequence length
mod 5 as we observed in Chalkboard #2

Using the mod function in MathCad, various sequence lengths of mod(A^m*v, n) can be checked by showing the initial vector (3,0,2) is calculated

$$\mathrm{mod}\left(A^{24} \cdot v, 5\right) = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \qquad \mathrm{mod}\left(A^{14} \cdot v, 4\right) = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathrm{mod}\left(A^{91}\cdot v,6\right)=\begin{pmatrix}3\\0\\2\end{pmatrix}$$

$$\mathrm{mod}\left(A^{48}\cdot v,7\right)=\begin{pmatrix}3\\0\\2\end{pmatrix}$$

Using Matrices to solve elliptic equations for z with g(z) = 0

g(z) = $z^3 + \alpha z + \beta$

$$\alpha := -1$$

$$\beta := -1$$

The Vandermonde matrix EV is defined as:

$$EV := \begin{pmatrix}1 & 1 & 1\\1 & \varepsilon & \varepsilon^2\\1 & \varepsilon^2 & \varepsilon^4\end{pmatrix}$$

where ε $\quad \varepsilon := \exp\left(\frac{2\cdot\pi\cdot i}{3}\right)$

Using the properties of the matrix EV and its inverse matrix;

$$E := \frac{1}{\sqrt{3}}\cdot EV \qquad EI := E^{-1}$$

where

$$EI\cdot E = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

The Vandermonde matrix is used to find solutions to the general elliptic equation. EV is called a Unitary Matrix which can diagonalize the elements of a Circulant Matrix

Definition: The 3x3 circulant matrix is associated with the vector (0,a,b) whose rows are shifted as follows:

$$C := \begin{pmatrix}0 & a & b\\b & 0 & a\\a & b & 0\end{pmatrix}$$

Remember in Chalkboard #1 a and b were defined for any value of α and β
Operate on C by taking the matrix products with EI and E:

$$EI \cdot C \cdot E = \begin{pmatrix} 1.324718 & 0 & 0 \\ 0 & -0.662359 + 0.56228i & 0 \\ 0 & 0 & -0.662359 - 0.56228i \end{pmatrix}$$

THIS GIVES A DIAGONAL MATRIX WITH THE SOLUTION
SET (EIGENVALUES) FOR THE ELLIPTIC EQUATION

NOTE: WITH A NON
CIRCULANT MATRIX (A)
THE MATRIX IS NOT
DIAGONALIZED

$$EI \cdot A \cdot E = \begin{pmatrix} 1.333 & -0.167 + 0.289i & -0.167 - 0.289i \\ -0.167 + 0.289i & -0.667 + 0.577i & 0.333 \\ -0.167 - 0.289i & 0.333 & -0.667 - 0.577i \end{pmatrix}$$

THE VANDERMONDE MATRIX IS INTRODUCED TO FIND SOLUTIONS TO
OTHER ELLIPTIC EQUATIONS AND TO GENERATE PERRIN LIKE SEQUENCES
WITH INTERESTING PROPERTIES.
THIS WILL BE COVERED IN A FUTURE CHALKBOARD.

The last topic of this chalkboard will be determinants of matrices.
Determinants are used to check if a square matrix (e.g.3X3) has
eigenvalue solutions (can be diagonalized).   If the determinant is zero
then a unique solution cannot be calculated.

Determinants will also be used in the next chalkboard to find the
number of fixed points in the finite sequence mod (m).
Many of these matrices (unimodular matrix) have a determinant of 1.
Some examples are shown below:
NOTE: the determinants of EV and EI are imaginary numbers.
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$|A| = 1$ $|I| = 1$ $|A^{13}| = 1$ $|C| = 1$

$|EV| = -5.196i$ $|EI| = i$ $|EI \cdot C \cdot E| = 1$

$$\left|C^{-1}\cdot 3\right| = 27$$

NOTE THAT THIS LAST
DETERMINANT IS 27*β^2
IT WILL BE SHOWN THAT THIS
IS AN INVARIANT FOR THE
ELLIPTIC EQUATION

The Trace (tr) of a diagonal matrix is the sum of the diagonal elements.
In the case of the diagonal matrix D, powers of D generate the Perrin Sequence S1(n)

$$D = \begin{pmatrix} 1.325 & 0 & 0 \\ 0 & -0.662 + 0.562i & 0 \\ 0 & 0 & -0.662 - 0.562i \end{pmatrix}$$

$\mathrm{tr}\left(D^0\right) = 3$

$\mathrm{tr}\left(D^7\right) = 7$

$\mathrm{tr}\left(D^1\right) = 0$

$\mathrm{tr}\left(D^8\right) = 10$

$\mathrm{tr}\left(D^2\right) = 2$

$\mathrm{tr}\left(D^{10}\right) = 17$

The Perrin Sequence mod (11) is shown in the graph at the start of this Chalkboard. If you look closely you may see multiple periodic behavior, Since 11 is a type 1 prime the sequence length is n^2 -1 = 120.   Can the mod 11 sequence also have a smaller sequence length?

Chalkboard #5 will discuss equivalence classes of the sequences and how to find hidden periodic behavior in a sequence!

RT