Chalkboard #5 Equivalence Classes

Questions:

1. Are there sequences equivalent to the Perrin sequence with a similar sequence length but not starting with the generator of S1(n) (3,0,2)?

2. Are there finite Perrin sequences with sequence length Ps(n) < P(n) or Ps(n)>P(n)?

This chalkboard examines properties of Ps(n) and S1(n). Let x1,x2,x3,x4,...xd be a sequence of integers with the following property:

x1+x2 = x4
x2+x3 = x5
.
.
x(d-3)+x(d-2) = x(d)

.Can we find a sequence of lenght d? (See reference 1 below)

Solution with Circulant matrices

Example: Let d=4 (a sequence which will cycle every 4 numbers)

Define the following dXd matrices $d := 4$

$$Wn := \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \qquad \Pi n := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad I1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A dX1 vector

$$x := \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}$$

Then $\begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} \equiv 0$ mod(n)

defines a Perrin circulant matrix Wn with 4 linear equations that must add to 0 mod(n)

Question: is there a mod(n) which gives a solution??

Notice that Πn is a forward permutation matrix acting on 1

$$\Pi n \cdot \Pi n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \Pi n^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

By combining I and powers of Πn we can obtain the circulant matrix Wn

$$I1 + \Pi n - \Pi n^3 = \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}$$

From our previous discussion of dXd Vandermonde matrices:

where ε is defined as

$$\varepsilon := \exp\left(\frac{2 \cdot \pi \cdot i}{d}\right)$$

$$EV := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^6 \\ 1 & \varepsilon^3 & \varepsilon^6 & \varepsilon^9 \end{pmatrix}$$

Using the properties of the matrix EV and its inverse matrix;

$$E := \frac{1}{\sqrt{d}} \cdot EV \qquad EI := E^{-1} \qquad E \cdot EI = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since Wn is a circulant matrix we obtain a diagonal matrix under operation E and the inverse EI.

$$Dn := EI \cdot Wn \cdot E$$

$$Dn = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+2i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-2i \end{pmatrix}$$

Trace $\qquad tr(Dn) = 4$

Determinant $\qquad |Dn| = 5$

NOTE: The trace recovers the length of the sequence (4) and the non-zero determine indicates there is a solution (Dn not zero)

We do not know what the vector of x's is but the determinant indicates its relative size.

In a tangent below I will explain how to find the vector x mod(n)

Given a sequence 1,2,4,3,

$$x = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix} \qquad \text{Then} \qquad Wn \cdot x = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 0 \end{pmatrix}$$

Since we are seeking a solution vector of zeros on the RHS let us take the solution mod(n) = mod(5) or mod (determinant of Wn)

$$mod(Wn \cdot x, 5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The finite Perrin sequence Ps(mod(5) is 1,2,4,3,1,2,4,3,.. with sequence length d=4!
..Note P(mod(5)) =24 with the Perrin sequence 3,0,2,3,2,0,0,2,0,2,2,4,.....

The above analysis results in large matrices for large d and becomes difficult to calculate with an accuracy to obtain integer values.

Fortunately there are several other methods of obtaining mod(n) solutions for the dXd matrix.

Calculation of mod(n) and d=4 with the Perrin Matrix

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Wnn := A^4 - I$$

$$|Wnn| = 5$$

Powers of the Perrin matrix can be used to obtain n= det(A^d-I)
Examples for d=1,2,3, 5, 6,7,8,9,10:

$$|A^1 - I| = 1 \qquad |A^2 - I| = 1 \qquad |A^3 - I| = 1$$

$$|A^5 - I| = 1 \qquad |A^6 - I| = 7 \qquad |A^7 - I| = 8$$

$$|A^8 - I| = 5 \qquad |A^9 - I| = 19 \qquad |A^{10} - I| = 11$$

The above sequence 0,1,1,1,5,1,7,8,5,19,11.. is found in OEIS A01945

Let (q,r,s) Perrin be the first 3- vector which generates a Perrin sequence. Then we can find finite sequence lengths (q,r,s) mod(n) with length d and mod (n=det(A^d-I))

Above we found (1,2,4) mod 5 has a sequence length of d=4. Similarly a sequence (q,r,s) mod 19 of length d=9 can be found.

Tangent: Solving for (q,r,s) with matrices,
Case: d=9, n=19

From above we know that

$A^9 * x = I * x$ *Find x= (q,r,s) mod(19)*

$$A^9 = \begin{pmatrix} 2 & 4 & 3 \\ 3 & 5 & 4 \\ 4 & 7 & 5 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

First define a new set of variables X=q/s and Y=r/s so s/s=1
This matrix represents the following series of equations in the vector
(X,Y,1):

*2*X+4*Y+3 = X*
*3*X+5*Y+4 =Y*
*4*X+7*Y+5 = 1*

Rearrange to get:

*X+4*Y = -3*
*3*X+4*Y = -4*
*4*X +7*Y = -4*

Add the second and 3rd equation to get the series to solve for X,Y

*X+4*Y = -3*
*7*X +11*Y = -8*

Cramer's rule is used to solve for q,r,s

$$Z := \begin{pmatrix} 1 & 4 \\ 7 & 11 \end{pmatrix}$$

$$|Z| = -17$$

$$X := \begin{pmatrix} -3 & 4 \\ -8 & 11 \end{pmatrix}$$

$$|X| = -1$$

$$Y := \begin{pmatrix} 1 & -3 \\ 7 & -8 \end{pmatrix} \qquad |Y| = 13$$

Convert to mod(19)
-17=2 mod(19) (Z)
-1 = 18 mod(19) (X)
13 = 13 mod(19) (Y)

Solution (q,r,s) = (18,13,2) mod(19)

Check: S1 mod(19) = ***18,13,2, 12,15,14,8,10,3,****18,13,2,....*

We can also find another solution since we can divide by s=2
*(q,r,s) = (18/2,13/2,2/2) = (9,16,1) Note 13=2*16 mod(19) so the second sequence is*
S1 mod(19) = ***9,16,1,6,17,7,4,5,11****,9,16,1,....*

We can define an equivalence class (d,n) of a sequence as those sequences with unique (q,r,s) vectors and having the same length d and modulus n.

In the above example (18,13,2) and (9,16,1) belong to the same class (9,19)

Also above (1,2,4) is the only element of the class (4,5).

In the next chalkboard we will determine the number of classes (d,n) equivalent to (3,0,2) for various prime values of n and find the number of sub-classes.

1. Reference for this Chalkboard: D.A. Coleman, C.J. Dugan, R.A. McEwen, C.A. Reiter, T.T. Tang, Periods of (q,r) Fibonnacci sequences and Elliptic curves, Fibonacci Quart. 44 (1) 2006 pp 59-70.

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