Chalkboard #6 Prime Classes, equivalence classes and exactly realizable sequences

In this chalkboard I will attempt to put together the concepts of prime classification introduced in Chalkboard #3, equilvalence classes and exactly realizable sequences[1].

1. I refer the reader to: Y.Puri and T. Ward "Arithmetic and growth of periodic orbits", J. Integer Sequences, vol 4. (2001).

First I will describe a third method for calculating the equivalence classes (d,n) introduced in the last Chalkboard.

The number of periodic points in the finite Perrin sequence (3,0,2) mod n is given by the magnitute of the the determinant $|(A^n - I)|$ where, given a 3x1 vector v

A^n*v = (μ+1)*v where (u+1)=λ are the eigenvalues of A^n and v is the corresponding eigenvector.

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We find that for the Perrin matrix A, there are three eigenvalues that are raised to the nth power. From Chalkboard #1 these eigen values are $r1^n$, $r2^n$ and $r3^n$ or $\lambda_{0,0}$, $\lambda_{1,0}$ $\lambda_{2,0}$ calculated below.

MathCad illustration for eigenvalues and eigenvectors of A

The Perrin matrix is and MathCad finds the eigenvalues and eigen vectors below:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\text{eigenvals}(A) = \begin{pmatrix} -0.66236 + 0.56228i \\ -0.66236 - 0.56228i \\ 1.32472 \end{pmatrix} \qquad \lambda := \text{eigenvals}(A)$$

$$\text{eigenvecs}(A) = \begin{pmatrix} 0.656 & 0.656 & 0.414 \\ -0.434 + 0.369i & -0.434 - 0.369i & 0.548 \\ 0.08 - 0.489i & 0.08 + 0.489i & 0.727 \end{pmatrix} \qquad V := \text{eigenvecs}(A)$$

Note: eignevalues and eigenvectors are complex numbers

$$\lambda_{0,0} = -0.662 + 0.562i$$

$$\lambda_{1,0} = -0.662 - 0.562i$$

$$\lambda_{2,0} = 1.325$$

EIGENVALUES OF A

$$x1 := V^{\langle 0 \rangle} \qquad x1 = \begin{pmatrix} 0.656 \\ -0.434 + 0.369i \\ 0.08 - 0.489i \end{pmatrix}$$

$$x2 := V^{\langle 1 \rangle} \qquad x2 = \begin{pmatrix} 0.656 \\ -0.434 - 0.369i \\ 0.08 + 0.489i \end{pmatrix}$$

$$x3 := V^{\langle 2 \rangle} \qquad x3 = \begin{pmatrix} 0.414 \\ 0.548 \\ 0.727 \end{pmatrix}$$

EIGENVECTORS OF A

Show that A*xi = λ*xi

$$A \cdot x1 = \begin{pmatrix} -0.434 + 0.369i \\ 0.08 - 0.489i \\ 0.221 + 0.369i \end{pmatrix} \qquad \lambda_{0,0} \cdot x1 = \begin{pmatrix} -0.434 + 0.369i \\ 0.08 - 0.489i \\ 0.221 + 0.369i \end{pmatrix}$$

$$A \cdot x2 = \begin{pmatrix} -0.434 - 0.369i \\ 0.08 + 0.489i \\ 0.221 - 0.369i \end{pmatrix} \qquad \lambda_{1,0} \cdot x2 = \begin{pmatrix} -0.434 - 0.369i \\ 0.08 + 0.489i \\ 0.221 - 0.369i \end{pmatrix}$$

$$A \cdot x3 = \begin{pmatrix} 0.548 \\ 0.727 \\ 0.962 \end{pmatrix} \qquad \lambda_{2,0} \cdot x3 = \begin{pmatrix} 0.548 \\ 0.727 \\ 0.962 \end{pmatrix}$$

The above analysis proves also that $(A^n - I)*v = (\lambda^n - 1)*v$

Also since the determinant of a matrix is equal to the product of its eigen values

we find an interesting relationship (shown for n=9)

$$\left| A^n - I \right| = 19$$

$$\left[(\lambda_{0,0})^n - 1 \right] \cdot \left[(\lambda_{1,0})^n - 1 \right] \cdot \left[(\lambda_{2,0})^n - 1 \right] \cdot 1 = 19$$

$$(\lambda_{0,0})^n + (\lambda_{1,0})^n + (\lambda_{2,0})^n - (\lambda_{0,0})^{-n} - (\lambda_{1,0})^{-n} - (\lambda_{2,0})^{-n} = 19$$

The last expression shows that the determinant of A^n-I is equal to the sum of the nth entry of the Perrin sequence minus the nth entry of the negative Perrin sequence explained in Chalkboard #1!

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THE EQUIVALENCE CLASS (d,n) is defined when n= S1(d)-SN1(d)

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Prime Classification and number of Prime equivalence classes.

1. Number of combinations of the 3 vector for modulus n.

It is simple to prove that given the n-1 numbers mod n there are n^3 combinations.

Examples mod(2). 2^3 = 8 combinations

(1,0,0), (0,0,1), (0,1,0), (1,0,1), (0,1,1), (1,1,1), (1,1,0) and (0,0,0)

Note that except for (0,0,0) all vectors are generated from (1,0,0) by taking powers of A.

$$C1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A \cdot C1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad A^2 \cdot C1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad A^3 \cdot C1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A^4 \cdot C1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad A^5 \cdot C1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad A^6 \cdot C1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2= 0 mod(2)

(0,0,0) is a vector which does not generate another vector.

We found that for type 2 primes (p) the finite sequence length is (p^2 +p +1)/b where b is a constant to be determined. This is also the number of 3-vectors in the finite sequence so the total number of equivalence classes times the sequence length of (q,r,s) should equal p^3. Since (0,0,0) is not part of the vectors (q,r,s), there are p^3-1 fixed points to account for. This means there are b*(p-1) equivalence classes for the type 2 primes (e.g. 2,3,13,29,31,...

b*(p-1)* (p^2 +p +1)/b = p^3 -1

For type 2 primes the number of equivalence classes of length (p^2 +p +1)/b is:

#((p^2 +p +1)/b, p^3) = b*(p-1)

Examples:

p=2: #(7, 8) = 1

p=3: #(13,27) =2

p=13 #(183, 2197) = 12

Similar arguments can be made for type 1 primes. However we can note from the values of

$|A^n\text{-}I|$ that there is also a class of length p-1 for these primes. Since the total number of vectors is p^3-1 and the normal length is(p2-1)/b

$a*(p)*(p^2\text{-}1)/a + b*(p\text{-}1)/a = p^3 \text{-}1$

For type 1 primes the number of equivalence classes of length (p^2 -1)/a and (p-1)/a is:

#((p^2 -1)/a, p^3) = a*(p) AND #((p -1)/a, p) = a

Examples:

p=5: #(24, 125) = 5 and #(4,5) = 1

p=7: #(48,343) =7 and #(6,7) =1

p=11 #(120, 1331) = 11 and #(10,11) = 1

There are 11 sequences of length 120 and 1 sequence of length 10 for S1 mod(11) for a total of 11*120 + 10 = 1331-1 = 1330 vectors

Type 3 primes are a sub-class of type 1 primes and we have shown that the perrin sequence length mod (p) is p-1 for this type of prime. Since we would also expect a sequence at p2-1 in this case we actually get another sequence length at p^2-1 – (p-1) = p^2-p. and

$(p+1)*(p-1) + p*(p^2-p) = p^3 - 1$

For type 3 primes the number of equivalence classes of length (p -1) and (p2-p) is:

#((p -1), p) = (p+1) AND #((p2 -p), p^3) = p

Examples:

p=23: #(22,23) = 24 and #(506,23^3) = 23

There are 24 sequences of length 22 and 23 sequence of length 506 for S1 mod(23) for a total of 24*22 + 23*506 = 12167-1 = 12166 vectors

Relating the number of elements in the class #(d,n) to OEIS table A001945

I will take another example for a type 1 prime to show how the prime classification is used to predict the number of equivalence classes.

The prime p=61 has a cyle length of 930 (see appendix to Chalkboard #2). This indicates the constant (b) is (61^2-1)/b = 930 or b = 4

Table A001945 show that for d=15 (sequence length 15), n = |A^{15} -I| = 61 so mod(61) will have b = 4 sequences of length 15.

*Checking: 4*61*930 + 4* (61-1)/4 = 61^3 - 1 = 226980*

Brief explanation of exactly realizable sequences

In all our examples with the Perrin Matrix A we are operating on vectors (q,r,s) to obtain new vectors. These vectors are in the space of integers. Also, all determinants of A are equal to unity. This defines a mapping called a toral automorphism on the 3 torus.

An exactly realizable sequence (*ER*) is a sequence which counts the number of points with a period n such that a set of points x is cyclic: $A^n *x = x$.

For a period of length n, the number of points is

$$\mathrm{Per}(n) \equiv \sum_{dn} (d \cdot \mathrm{Orb}(n))$$

where dn is the set of all divisors of n and the sum is over all divisors.

The number of orbits of period n is

$$\mathrm{Orb}(d) \equiv \frac{1}{d} \cdot \left[\sum_{dn} (\mu(dn) \cdot \mathrm{Per}(n)) \right]$$

where μ(dn) is the Mobius function (see OEIS A008683 which is a function dn with values of -1, 0 or 1.

We can check that the sequence of (d,n) in OEIS A001945 is exactly realizable:

The 24th element is 875 or (24,875). The divisors dn of d=24 are 1,2,3,4,6,8,12 and 24

Per(n)	dn	μ(dn)	Per(n)	μ(dn)*Per(n)	Orb(n)
Per(24)	1	1	875	875	calculate
Per(12)	2	-1	35	-35	2
Per(8)	3	-1	5	-5	0
Per(6)	4	0	7	0	1
Per(4)	5	1	5	5	1
Per(3)	8	0	1	0	0
Per(2)	12	0	1	0	0
Per(1)	24	0	1	0	1
			SUM	840	

Orb(24) = 840/24 = 35

Per(24) = 24*35 + 12*2+ 8*0 +6*1+ 4*1 + 3*0 + 2*0 1*1 = 875

For p = 5 a class 1 prime #(4,5) = 1 and #(24,5^3) = 5 . Note that 125 divides 875.

The generators for the 5 orbits in the class (24, 125) are (3,0,2), (0,0,1), (0,0,3), (0,0,4) and (1,2,3). All vectors from these five sequences give a total of 24*5 =120 vectors. The remaining vectors are generated from the class (4,5) with generator (1,2,4). The last element is (0,0,0) to account for the 5^3 vectors.

From the above analysis the following conjectures are made.

Let Wn be the value of the sequence length P(n) for which S1(P(n) – SN1(P(n) = 0 mod (w)

Wn = {P(n) | S1(P(n) – SN1(P(n) = 0 mod (w)}

The value of w is given in the conjecture:

Conjecture: The constants a, b, and c defined for type 1, type 2 and type 3 primes are given as:

For Type 1 and Type 2 primes p = n

$(n^2-1)/Wn = a$ and $(n^2+n+1)/Wn = b$ where $w = n^3$.

For type 3 primes p = n

(n-1)/Wn = c where w = n

For the first 10 type 1 primes (5,7,11,17,19,37,43,53,61,67) the values of a are (1,1,1,1,2,1,8,2,4,1)

For the first 10 type 2 primes (2,3,13,29,31,41,47,71,73,127) the values of b are (1,1,1,2,2,2,2,2,2,2)

For the first 10 type 3 primes (23,59,101,167,173,223,271,307,317,347) the values of c are
(1,1,1,1,1,2,1,1,1,2)

In the first 1000 numbers the largest value of a is 480 for p = 911 P(n) = 1729, b is 7 for p= 739 P(n) = 78123 and c is 2; several occur, the first at p = 223 P(n) = 111.

Note: If anyone has a solution to finding the sequences for a, b, and c above please leave a comment!

I will discuss in the next Chalkboard equivalence classes for which n is not a prime (e.g. n is divisible by primes)

RT