Chalkboard #7 Higher Order Perrin Sequences

The sequences generated from the 3rd order equation $x^3 - x - 1 = 0$ have been discussed in the previous chalkboards. In this board I will discuss sequences generated from the 4th and 5th order equations:

$x^4 - x - 1 = 0$
$x^5 - x - 1 = 0$

MathCad is used to find the 4 roots (2 real and 2 complex) to the 4th order equation:

$$f(x) := x^4 - x - 1$$

$$r0 = -0.248 + 1.034i$$

$$r1 = -0.248 - 1.034i$$

$$r3 = -0.7245$$

$$r4 = 1.2207$$

The 4 solutions provide the sequences of 4th order: S4 and SN4.
The series are calculated below by input of term n1

$$n1 = 7$$

$$S4 := r0^{n1} + r1^{n1} + r3^{n1} + r4^{n1}$$

$$SN4 := r0^{-n1} + r1^{-n1} + r3^{-n1} + r4^{-n1}$$

$$S4 = 7$$

$$SN4 = -8$$

The order 4 sequence S4 is OEIS A050443

4,0,0,3,4,0,3,7,4,3,10,...

a(n) = a(n-3) + a(n-4)

The inverse sequence SN4 is

4,-1,1,-1,5,-6,7,-8,13,-19,26..

an(n) = an(n-4) - an(n-1)..

The order 4 matrices are

$$A4 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad An4 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As in the Perrin sequence we can find the number of fixed points at each finite sequence length. First find solutions to the determinant of ($A4^n$ -I). Consider the 9th term

The eigen value solution to the circulant matrix

$I + \Pi n - \Pi n^4$

for a 9 cycle (this gives 9 eigenvalues)

$ex4 = 9$

$m4 := ex4$

$j := 1 .. m4$

$$\lambda_j := 1 + \exp\left(2 \cdot \pi i \cdot \frac{j}{m4}\right) - \exp\left(8 \cdot \pi i \cdot \frac{j}{m4}\right)$$

$$Per4 := (-1)^{m4-1} \cdot \prod_{j=1}^{m4} \lambda_j$$

$Per4 = 37$

This solution is the 4th order case which is similar to the number of fixed points from OEIS A001945

Remember that the ninth term was 19 from OEIS A001945

As in the 3 rd order Perrin sequence we can also calculate Per4 from the 4x4 matrices A4 and I above. The sign is corrected using powers $(-1)^n$

$$\left|A4^{ex4} - I\right| \cdot (-1)^{m4} = 37$$

Interpretation of result: The Per4th term is a 37 which is
the smallest mod of cycle n=9 for a 4th order Perrin
sequence mod 37

$$A4^{ex4} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix} \qquad An4^{ex4} = \begin{pmatrix} 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 5 \\ 5 & 4 & 3 & 7 \\ 7 & 5 & 4 & 10 \end{pmatrix}$$

Use the sequence (q,r,s,t) generated from (36,4,21,27) From the vector x24

$$x24 := \begin{pmatrix} 36 \\ 4 \\ 21 \\ 27 \end{pmatrix}$$

$$aa24 := \mathrm{mod}\left(A4^{9} \cdot x24, 37\right) \qquad aa24 = \begin{pmatrix} 36 \\ 4 \\ 21 \\ 27 \end{pmatrix}$$

The vector of aa24 proves the interpretation given
a 9 cycle:for mod(37)

36,4,21,27, 3,25,11,30,28,36,4,21,27,....

There are 3 other sequences of length 9 (multiples of the above sequence mod(37) suggesting that the Per4(n) -1 vectors (q,r,s,t) are in an equivalence class of mod (n).

The sequence Per4(n), first 20 terms

0,1,1,1,1,11,7,1,17,37,11,23,91,79,29,176,289,137,259,761,671

Note that the smallest sequence length of 5 is generated mod(11). Unlike the order 3 sequence there is no sequence of length 4 for any mod(n)

The sequence appears to be exactly realizable as all primes are divisible by
Per4(n)-1 e.g. n=17 Per4(n) = 137-1 =136 = 17*8

It can easily be determined in the table below that the first few primes do not follow a simple
expression for sequence length.(see details below)
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TABLE I

Prime p	Cycle length	Formula
2	15	2^4 -1
3	80	3^4 -1
5	312	$(5^4$ -1)/2
7	342	7^3 -1
11	1330	11^3 -1
13	2196	13^3 -1
17	1632	$(17^2$ -1)*17/3

Closed form for calculating Per4(n). It would be convenient to calculate Per4(n) from the sequences S4 and SN4 as in the 3rd order case. Note that like the 3rd order case the product of the roots is negative 1 similar to the determinant of A4 indicating the map is a toral automorphism [det A = +/- 1]..

$$r0 \cdot r1 \cdot r3 \cdot r4 = -1$$

$$|A4| = -1$$

It can be shown that Per4(n) can be calculated from S4(n), SN4(n), $1+(-1)^n$, and a cross product expression

$XC(n) = (S4(n)^2 - S4(2n))/2$

Example: Compute Per4(15)

S4(15) = 18
SN4(15) = 126
$1+(-1)^{15} = 0$
S4(30) = 388

$Per4(15) = 18 +126 + 0 + (18^2 - 388)/2 = 176$

$Per4(n) = S4(n) + SN4(n) +XC(n)+1+(-1)^n$

Tangent: I will give results for the Perrin sequence of 5th order for a closed expression for Per5(n)
In this case there are two cross terms

$X2C5 = (S5(n)^2 - S5(2n))/2$

X3C5 = (S5(n)*X2C5 -S5(n)*S5(2*n) +S5(3*n))/3

where S5(n) is the nth term of the 5th order perrin sequence and
SN5(n) is the nth term of the inverse fifth order perrin sequence.

Then
:
Per5(n) = S5(n) + SN5(n) -X2C5 + X3C5

The order 5 sequence S5 is OEIS A087935

5,0,0,0, 4,5,0,0,4,9,5,0,4,...

a(n) = a(n-5) + a(n-4)

The inverse sequence SN5 is

5,-1,1,-1,1,4,-5,6,-7,8,-4,-1,7..

an(n) = an(n-5) - an(n-1)..

Regarding TABLE I: In the sequence Per4(n) there are prime patterns that differ from the Per3(n) sequence. Note that based on the equivalence classes mod (p) of length (d,n) it is required that the total number of 4 vectors (q,r,s,t) is p^4.

Since there are 2 orbits for mod(5) the total $5^4 = 2*(5^4 -1)/2 + 1 = 625$. For primes 7,11, and 13 note that Per4(n) contains sequence lengths at Per4(p) – 1. { (6,7), (10,11), (12,91=13*7). For p = 17 there are two classes at (16, 289 = 17*17), (8,17) so

$17^4 = 17*3*(17^2 -1)*17/3 + 17*16 +2*8 + 1 = 83521$

Other prime classes included in Per4(n) are (14,29), (22,1541= 23*67). Note that for d = 18, 19 does not divide 259 so p = 19 is of a similar class to p = 2,3 and 5

0,1,1,1,1,11,7,1,17,37,11,23,91,79,29,176,289,137,259,761,671,463,1541,.

Conjecture: Primes of the form $p*2^k + 1$ where p is an odd prime (e.g. 7,11,13,23,29,) are a type 1 prime which splits primes of sequence length $(p^3 -1)$

Based on the 4th and 5th and higher order equations the nth term is dependent on the 2n, 3n, 4n….etc terms of higher order Perrin sequences. As the order increases an infinite number of "future" terms are required to define the "present". In this case the present is deterministic.
Can it be proven that higher order Perrin sequences do not become chaotic for large primes?
Chaotic sequences mod(p) destroy the deterministic behavior of an exactly realizable sequence!

Next Chalkboard:

Other 3rd order Perrin sequences
and Invarients

RT