CHALKBOARD #10 Dynamical Zeta Functions

Baake, M, Lau,E, and Paskunas, V, A Note on the Dynamical Zeta Function of General Toral Endomorphisms, arXiv:0810.1855v2 13 Jan 2009.

The Artin-Mazur zeta function relates the periods and orbit count. Previously, I discussed the exactly realizable sequences which were integer sequences that counted the orbits and fixed points.

For a period of length n, the number of points is

$$\mathrm{Per}(n) \equiv \sum_{dn} (d \cdot \mathrm{Orb}(n))$$

where d is a divisor or n and the sum is over all divisors.

The number of orbits of period n is

$$\mathrm{Orb}(d) \equiv \frac{1}{d} \cdot \left[\sum_{dn} \left(\mu(dn) \cdot \mathrm{Per}(n) \right) \right]$$

where $\mu(dn)$ is the Mobius function (see OEIS A008683 which is a function dn with values of -1, 0 or 1. These equations are true for *quasihyperbolic* matrices, or matrices with eigenvalues not equal to imaginary roots of unity (e.g $i^n = 1$ where i is imaginary)

Using orbits and Period lengths from OEIS A001945,and A060169, the Artin-Mazur zeta is defined as:

$j := 1 .. 40$

$orb_j :=$	$Per_j :=$
1	1
0	1
0	1
1	5
0	1
1	7
1	8
0	5
2	19
1	11
2	23
2	35
2	27
4	64
4	61
5	85
8	137
6	133
12	229
13	275
16	344
23	529
26	599
35	875
46	1151
54	1431
76	2071
89	2560
120	3481
154	4697
192	5953
255	8245
322	10649
411	14111
544	19048
679	24605
898	33227
1145	43739
1476	57591
1925	77275

$$LH := \exp\left[\sum_j \left(\frac{Per_j}{j} \cdot z^j\right)\right]$$

A Riemann summation

$$RH := \prod_j \left(1 - z^j\right)^{-orb_j}$$

An Euler product

$$LH - RH = 0$$

or taking the log of both sides of LH and RH

$$LH2 := \sum_j \left(\frac{Per_j}{j} \cdot z^j\right)$$

$$RH2 := \sum_j \left(orb_j \cdot \ln\left(1 - z^j\right)\right)$$

$$LH2 + RH2 = 0$$

The Generating Function for the series A001945 was found in Chalkboard #9.
In the equations below u is a real number.

$$GP(u) := \frac{u\cdot\left(\left(1 + 2\cdot u + u^2 + 2\cdot u^3 + u^4\right)\right)}{\left(u^3 - u - 1\right)\cdot\left(u^3 + u^2 - 1\right)}$$

where

$$GH2(u) := \sum_j \left(Per_j\cdot u^j\right)$$

It is interesting that the Generating function and Zeta function
are related by an integral transformation.

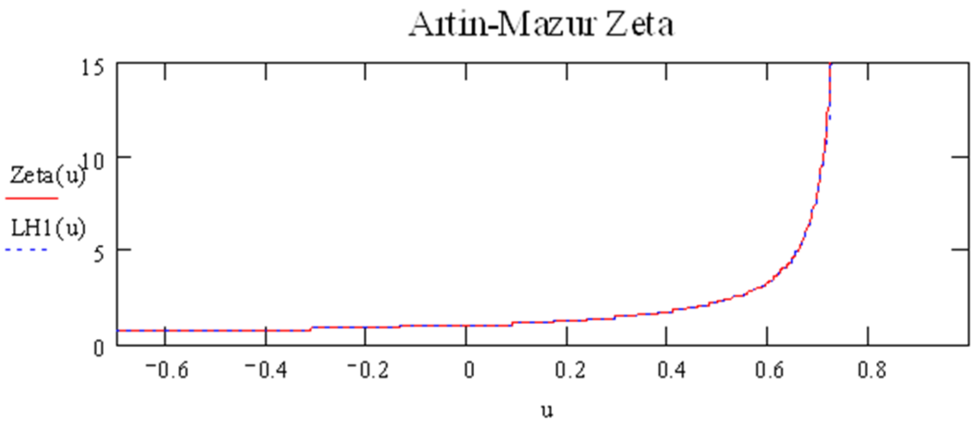
Find the Zeta Function
from the Generating
function.

$$IX1(u) := \int_0^u \frac{GP(u)}{u}\, du$$

The integral of the generating function over u is equal to the Riemann sum of the period sequence!

$$Zeta(u) := \exp(IX1(u))$$

$$LH1(u) := \exp\left[\sum_j \left(\frac{Per_j}{j} \cdot u^j\right)\right]$$



We can also reverse the process by obtaining the Generating Function from the Artin Zeta Function by using the derivative of the Zeta.

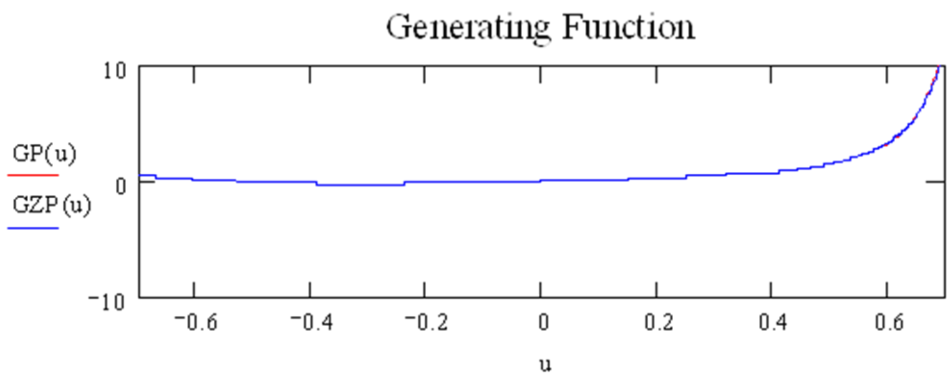
$$LH1(u) := \exp\left[\sum_j \left(\frac{Per_j}{j} \cdot u^j\right)\right]$$

$$DLH1(u) := \frac{d}{du} LH1(u)$$

$$GZP(u) := \frac{u \cdot DLH1(u)}{LH1(u)}$$

This is the rational Generating function GP(u)!!

$$GP(u) := \frac{u \cdot \left(\left(1 + 2 \cdot u + u^2 + 2 \cdot u^3 + u^4\right)\right)}{\left(u^3 - u - 1\right) \cdot \left(u^3 + u^2 - 1\right)}$$



It is difficult (if not impossible) to find the Zeta function as a function of u only. For certain types of matrices called the adjacency matrix of a graph the Ihara Zeta can be used. However, this applies only to connected graphs which are backtrackless. Good graphs follow the vertices (v1,v2,v3,v4,v5...vn). A backtrackless graph does not allow paths which reverse direction such that (v1,v2,v3,v2,v3,v4....) If the Perrin matrix is viewed as an adjacency matrix we find that v1,v2,v3,v2,v3.. is allowed so the graph does not conform to the Ihara Zeta.

As an example of a graph that is backtrackless.   The Laplacian is defined as the the functionn Δ(u) where AK4 is the adjacency matrix of graph K4 (vertices form a triangular pyramid), ID is the identity matrix, Q is a diagonal matrix diag(degee-1) where degree is the number of edges from a vertex (3 in this case) and |Δ| is the determinant of the Laplacian. χ is equal to the number of edges – number of vertices = |6|-|4| = 2

Ihara Zeta for K4

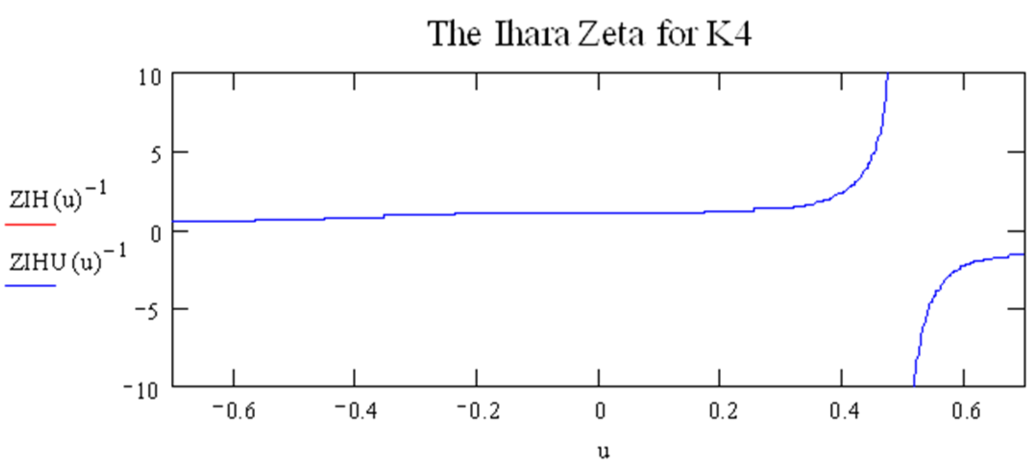
$$AK4 := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad Q := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad ID := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Delta(u) := ID - AK4\cdot u + Q\cdot u^2 \qquad \chi := 2$$

$$ZIH(u) := \left(1-u^2\right)^{\chi}\cdot\left|\Delta(u)\right|$$

$$ZIHU(u) := \left(1-u^2\right)^2\cdot(1-u)\cdot(1-2\cdot u)\cdot\left(1+u+2\cdot u^2\right)^3$$

ZIH(u) is the inverse of the Ihara Zeta which is a function of u. The function of u, ZIHU(u) is found by using well known methods of solving determinants.



Another interesting power series identity which applies to all Zeta functions is shown below:

Given the identity matrix and the Perrin matrix A1

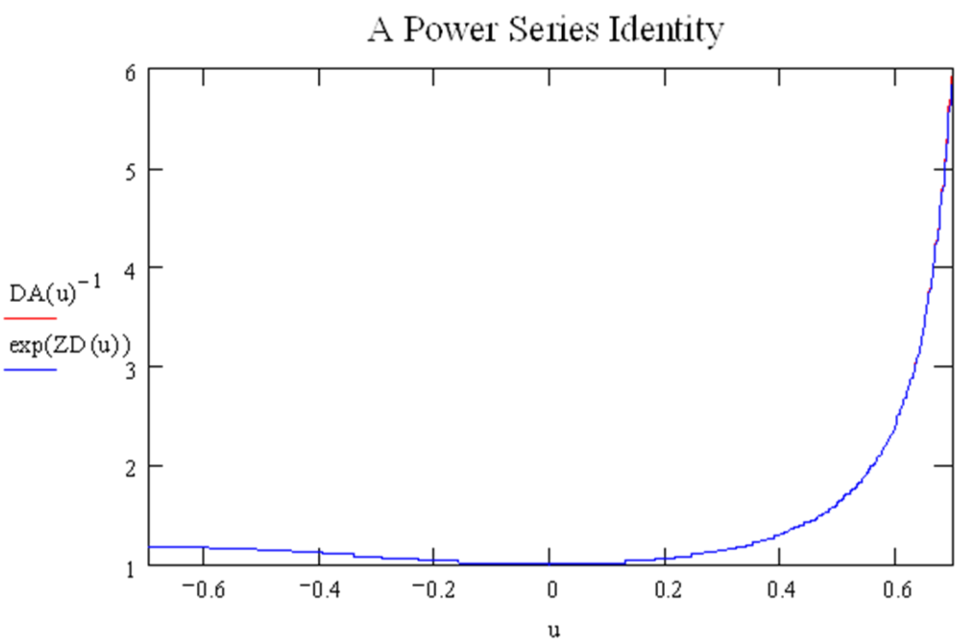
$$A1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$DA(u) := \left| I - u \cdot A1 \right|$$

$$ZD(u) := \sum_j \left(\frac{tr\left(A1^j\right)}{j} \cdot u^j \right)$$

1/DA(u) = ZD(u)

where $tr(A1^j)$ is the trace of the power matrix A1.
This seems to be true even for paths which backtrack.



As a final example of a Zeta function we calculate the orbit counts for the (4,5) Elliptic equation presented in Chalkboard #8

$orb4_j :=$

8
4
40
196
448
3220
9680
51300
218880
944384
4603600
19778580
95379680
433720040

$Per4_j :=$

8
16
128
800
2248
19456
67768
411200
1970048
9446096
50639608
237363200
1239935848
6072148336

$$LH4 := \exp\left[\sum_j \left(\frac{Per4_j}{j} \cdot z^j\right)\right]$$

$$RH4 := \prod_j \left(1 - z^j\right)^{-orb4_j}$$

$$LH4 - RH4 = 0$$

$$LH24 := \sum_j \left(\frac{Per4_j}{j} \cdot z^j\right) \qquad RH24 := \sum_j \left(orb4_j \cdot \ln\left(1 - z^j\right)\right)$$

$$LH24 + RH24 = 0$$

The Generating function for (4,5) as given previously

$\alpha := 4$ $\qquad \beta := 5$

$f0 := 3$ $\qquad f1 := -\alpha$ $\qquad f2 := \alpha^2$

$$S45(u) := \frac{3 - \alpha \cdot u^2}{1 - \alpha \cdot u^2 - \beta \cdot u^3}$$

$$SN45N(u) := \left[\frac{f0 + (f0 \cdot \alpha + f1) \cdot u + (f2 + f1 \cdot \alpha) \cdot u^2}{1 + \alpha \cdot u - \beta \cdot u^3}\right]$$

$$SN45D(u) := \frac{1}{1 - \beta \cdot u}$$

$$ONE(u) := \frac{1}{1 - u}$$

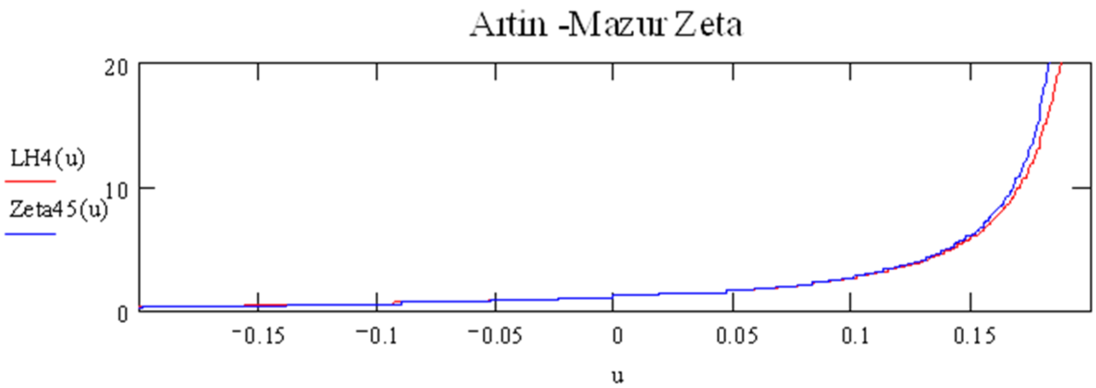
$$GF45(u) := S45(u) - SN45N(u) + SN45D(u) - ONE(u)$$

$$IX45(u) := \int_0^u \frac{GF45(u)}{u} \, du$$

Zeta Function from generating function

$$Zeta45(u) := \exp(IX45(u))$$

$$LH4(u) := \exp\left[\sum_j \left(\frac{Per4_j}{j} \cdot u^j\right)\right]$$

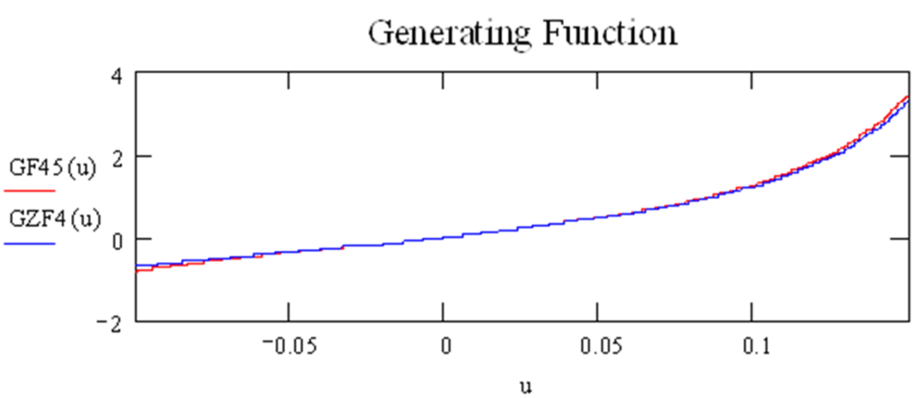


$$LH4(u) := \exp\left[\sum_j \left(\frac{Per4_j}{j} \cdot u^j\right)\right]$$

Generating Function from Zeta function

$$DLH4(u) := \frac{d}{du} LH4(u)$$

$$GZF4(u) := u \cdot \frac{DLH4(u)}{LH4(u)}$$



A reference for Zeta functions and trace formula is:
AudreyTerras, Fourier Analysis on Finite Groups and Applications,
Chapter 24, Cambridge University Press, 1999.

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The next board will discuss another application of the Perrin Sequence called maximal independent sets (MIS) of the n-cycle graph.   The number of MISs is the nth term of the Perrin Sequence!

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