# Chalkboard #11 Maximal Independent Sets, Cycles, and Spanning Trees

In graph theory an independent set is a set of vertices in a graph in which no two vertices are adjacent so no edge connects these two vertices. The maximal independent set (MIS) is the largest set of independent sets for a graph. An MIS is not a subset of any independent set.

The number of maximal independent sets for an n-cycle or an n-gon graph can be found from the Perrin sequence of numbers.

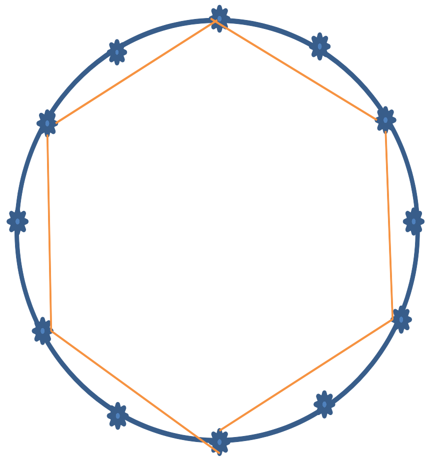
Reference: R. Bisdorff and JL. Marichal, Counting non-isomorphic maximal independent sets of the n-cycle graph, J. of Integer Sequences, Vol. 11 (2008).

Every graph contains at most $3^{(n/3)}$ maximal independent sets but many graphs such as the n-cycle have fewer. The n-vertex cycle graph having 12 vertices has P(12) = 29 maximal independent sets. This is far fewer than the $3^{(4)} = 81$ possible MISs for graphs with 12 vertices.

Below is an example of a familiar cycle dodecagon clockface (n=12 vertices). As described below there are 4 non-isomorphic MISs (red graphs). For each non-isomorphic graph there are isomorphic rotations on the n-cycle. The number of orbits for each non-isomorphic MIS can be calculated by a mobius transform of the Perrin sequence.

The non-isomorphic MIS are also formed from the combinatorial sums of the numbers 2 and 3. Each sum adds to n (12).

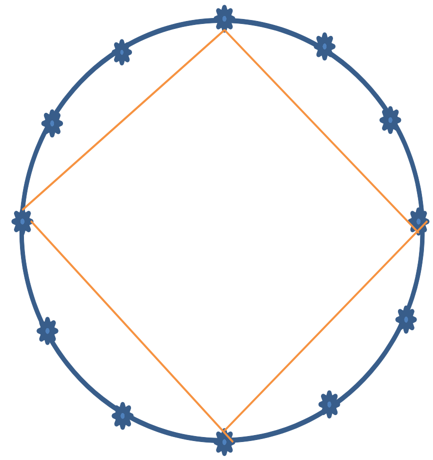
Note that maximal sets with non adjacent vertices at i+2 or i+3 are only allowed.



2 orbits

222222

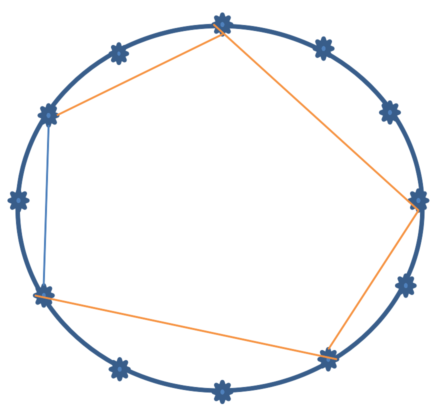
| | |
|---|---|
| hours | 12,2,4,6,8,10 |
| hours | 1,3,5,7,9,11 |



3 orbits

3333

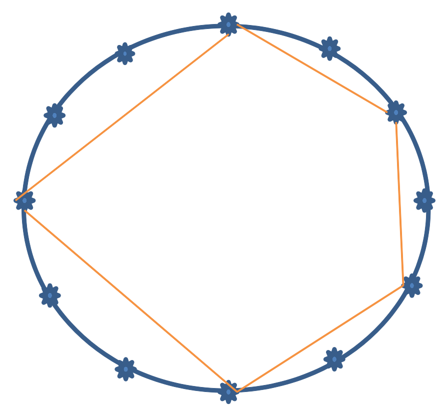
| | |
|---|---|
| hours | 12,3,6,9 |
| hours | 1,4,7,10 |
| hours | 2,5,8,11 |



12 orbits

32322

| | |
|---|---|
| hours | 12,3,5,8,10 |
| hours | 1,4,6,9,11 |
| hours | 2,5,7,10,12 |
| hours | 3,6,8,11,1 |
| hours | 4,7,9,12,2 |
| hours | 5,8,10,1,3 |
| hours | 6,9,11,2,4 |
| hours | 7,10,12,3,5 |
| hours | 8,11,1,4,6 |
| hours | 9,12,2,5,7 |
| hours | 10,1,3,6,8 |
| hours | 11,2,4,7,9 |



12 orbits

22233

| | |
|---|---|
| hours | 12,2,4,6,9 |
| hours | 1,3,5,7,10 |
| hours | 2,4,6,8,11 |
| hours | 3,5,7,9,12 |
| hours | 4,6,8,10,1 |
| hours | 5,7,9,11,2 |
| hours | 6,8,10,12,3 |
| hours | 7,9,11,1,4 |
| hours | 8,10,12,2,5 |
| hours | 9,11,1,3,6 |
| hours | 10,12,2,4,7 |
| hours | 11,1,3,5,8 |

If f(d) = $d^{th}$ term of the Perrin sequence and g(n/d) is the mobius sequence $\mu$, then f(n) is the mobius transform of the orbit count of the Perrin sequence.

The number of isomorphic orbits is d * orbit

The number of non isomorphic orbits of size d is orb(d) where d divides n

The convolution product of two sequences is given by

(f*g)(n) = $\sum_{dn} \left( f(d) \cdot g\left(\frac{n}{d}\right) \right)$ when dn is d divides n

The jth term of the Perrin sequence P and the orbit (orb) is defined for the first 25 terms

orb = 1/n* $\sum_{dn} \left( P(d) \cdot \mu\left(\frac{n}{d}\right) \right)$

$P_j$ = Perrin sequence

$P_j :=$

| |
|---|
| 0 |
| 2 |
| 3 |
| 2 |
| 5 |
| 5 |
| 7 |
| 10 |
| 12 |
| 17 |
| 22 |
| 29 |
| 39 |
| 51 |
| 68 |
| 90 |
| 119 |
| 158 |
| 209 |
| 277 |
| 367 |
| 486 |
| 644 |
| 853 |
| 1130 |

$orb_j :=$

| |
|---|
| 0 |
| 1 |
| 1 |
| 0 |
| 1 |
| 0 |
| 1 |
| 1 |
| 1 |
| 1 |
| 2 |
| 2 |
| 3 |
| 3 |
| 4 |
| 5 |
| 7 |
| 8 |
| 11 |
| 13 |
| 17 |
| 21 |
| 28 |
| 34 |
| 45 |

As found in Chalkbord #10 there is a Artin-Mazur zeta function which relates these two sequences:

$$LH(u) := \sum_j \frac{(P_j) \cdot u^j}{j}$$

$$RH(u) := \left[ \prod_j \left(1 - u^j\right)^{-orb_j} \right]$$

$$ExLH(u) := \exp(LH(u))$$

orb is given by OEIS A113788

The total number of unlabeled MISs of the n-cycle is also the number of cyclic combinations of 2 and 3 which add to give n. This number is calculated from the euler totient tranform . The euler totient is the number of non divisors of n less than n. For example the totient $\phi$ (12) = 4 since 1,5,7,11 are the only numbers less than 12 having no common divisor.

The jth  term of the Perrin sequence P and the sigma orbit (orb$\sigma$) is defined for the first 25 terms

orb$\sigma$ = 1/n* $\sum_{dn} \left( P(d) \cdot \phi\left(\frac{n}{d}\right) \right)$

$\phi_j$ =

| |
|---|
| 1 |
| 1 |
| 2 |
| 2 |
| 4 |
| 2 |
| 6 |
| 4 |
| 6 |
| 4 |
| 10 |
| 4 |
| 12 |
| 6 |
| 8 |
| 8 |
| 16 |
| 6 |
| 18 |
| 8 |
| 12 |
| 10 |
| 22 |
| 8 |
| 20 |

orb$\sigma_j$ :=

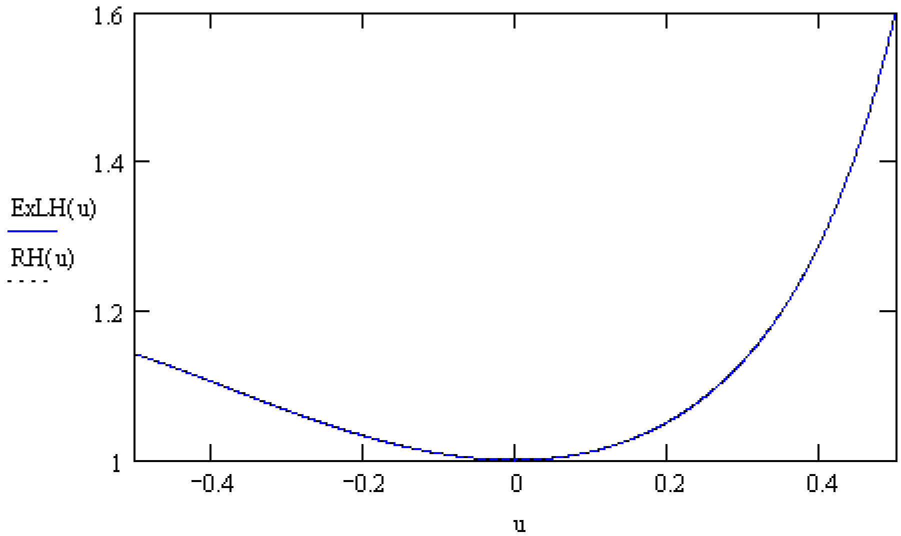
| |
|---|
| 0 |
| 1 |
| 1 |
| 1 |
| 1 |
| 2 |
| 1 |
| 2 |
| 2 |
| 3 |
| 2 |
| 4 |
| 3 |
| 5 |
| 6 |
| 7 |
| 7 |
| 11 |
| 11 |
| 16 |
| 19 |
| 24 |
| 28 |
| 39 |
| 46 |
| |

Example orb$\sigma$(14) =
1/14*(P(14)*$\phi$(1)+:P(7)*$\phi$(2)+P(2)*$\phi$(7)+P(1)*$\phi$(14) =
1/14*(51*1+7(1)+2*(6)+0(6) = 70/14 = 5

Ther are 5 cyclic combinations of 2 and 3 adding to 14:

2+2+2+2+2+2+2
2+2+3+2+2+3
2+2+2+2+3+3
2+3+3+3+3
3+2+2+2+3+2

The Perrin sequence  and orb=A113788 are related by the Zeta function




Bisdorf describes other sequence used for finding MISs of cycle graphs.  Two related sequence which I will only mention are the Padovan sequence (OEIS A000931) 0,1,1,1,2,2,3,4,5,7,9,… Pd(n)= Pd(n-2) + Pd(n-3) which we found in an earlier board to be an equivalence class of period lengths of mod(n). Also r(n)= r(n-4) + r(n-6) (OEIS A127682) is a sequence derived from the Padovan sequence giving the number of non-isomorphic MISs of the n-cycle having at least 1 symmetry axis. In the example of the dodecagon above all 4 non-isomorphic classes have at least one symmetry axis.(lines from 12 hr to 6 hr in top graphs, and 10hr to 4hr and 9 hr to 3 hr in bottom graphs)

Given previously I described  the generating function for the Perrin sequence. As an exercise the Zeta function above is used to find the Generating function.

$$GP(u) := \frac{3 - u^2}{1 - u^2 - u^3}$$

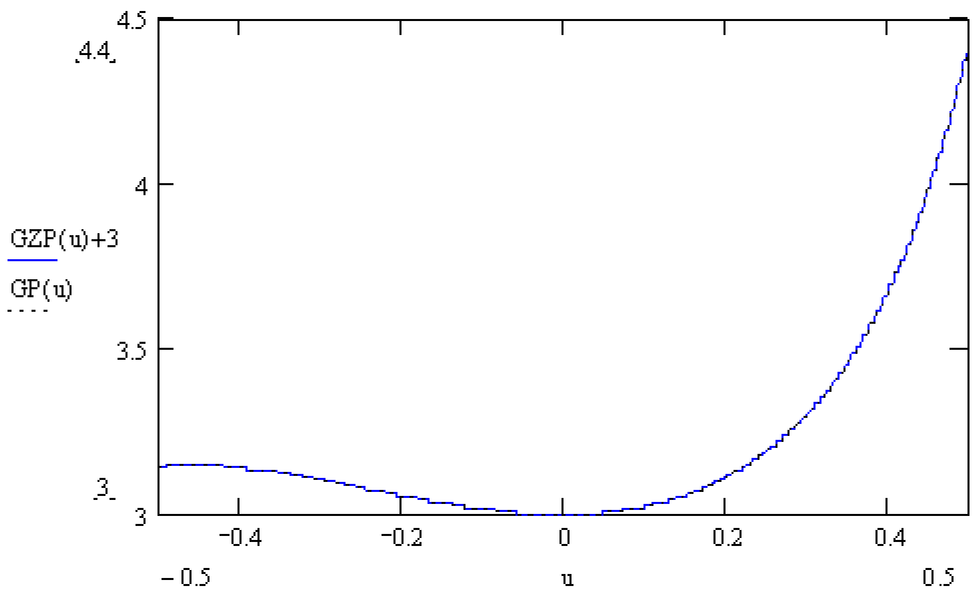
$$LH1(u) := \exp\left[\sum_j \left(\frac{P_j}{j} \cdot u^j\right)\right]$$

$$DLH1(u) := \frac{d}{du} LH1(u)$$

The derivative of the zeta function is shown to be the generating function GP(u)

$$GZP(u) := \frac{u \cdot DLH1(u)}{LH1(u)}$$

Generating function for Perrin sequence  is Zeta +3



Do any of the other related Perrin sequences count cycles or independent sets?

The answer is "yes". Consider the (2,1) perrin sequence on $x^3$-2x-1

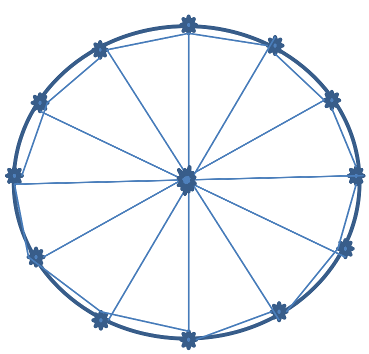
P21 = 3,0,4,3,8,10,19,28,48,75,124,198...

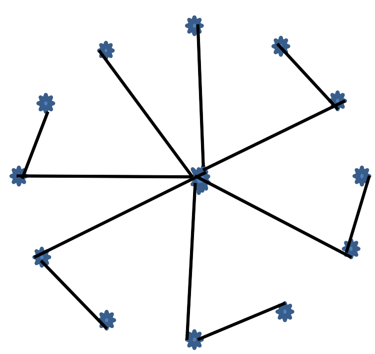
It is known that the Lucas sequence (OEIS A000032) $L_n$ counts the number of spanning trees of a labeled wheel on n+1 points. A wheel is an n-cycle with spokes. See example below for the dodecagon with the added point (n+1=13)

The Lucas number is calculated from the sequence 2,1,3,4,7,11,…

A spanning tree is a graph all points connected but no cycles.

Wheel with n+1 = 13 points and a sample spanning tree





$L_0 := 2$

$L_1 := 1$

$L_i := L_{i-2} + L_{i-1}$

$P21_i := L_i + (-1)^i$

The number of spanning trees in a labeled wheel on n+1 points is:

$NST_i := P21_{2 \cdot i} - 3$

$L_i =$

| |
|---|
| 1 |
| 3 |
| 4 |
| 7 |
| 11 |
| 18 |
| 29 |
| 47 |
| 76 |
| 123 |
| 199 |
| 322 |
| 521 |
| 843 |
| 1364 |

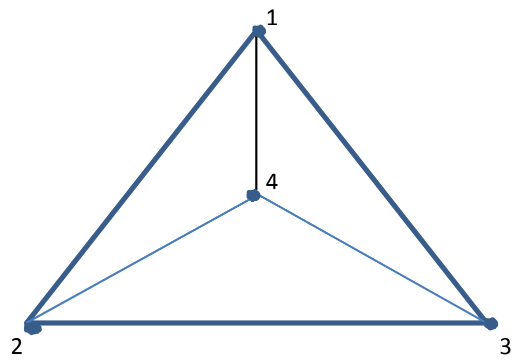
$P21_i =$

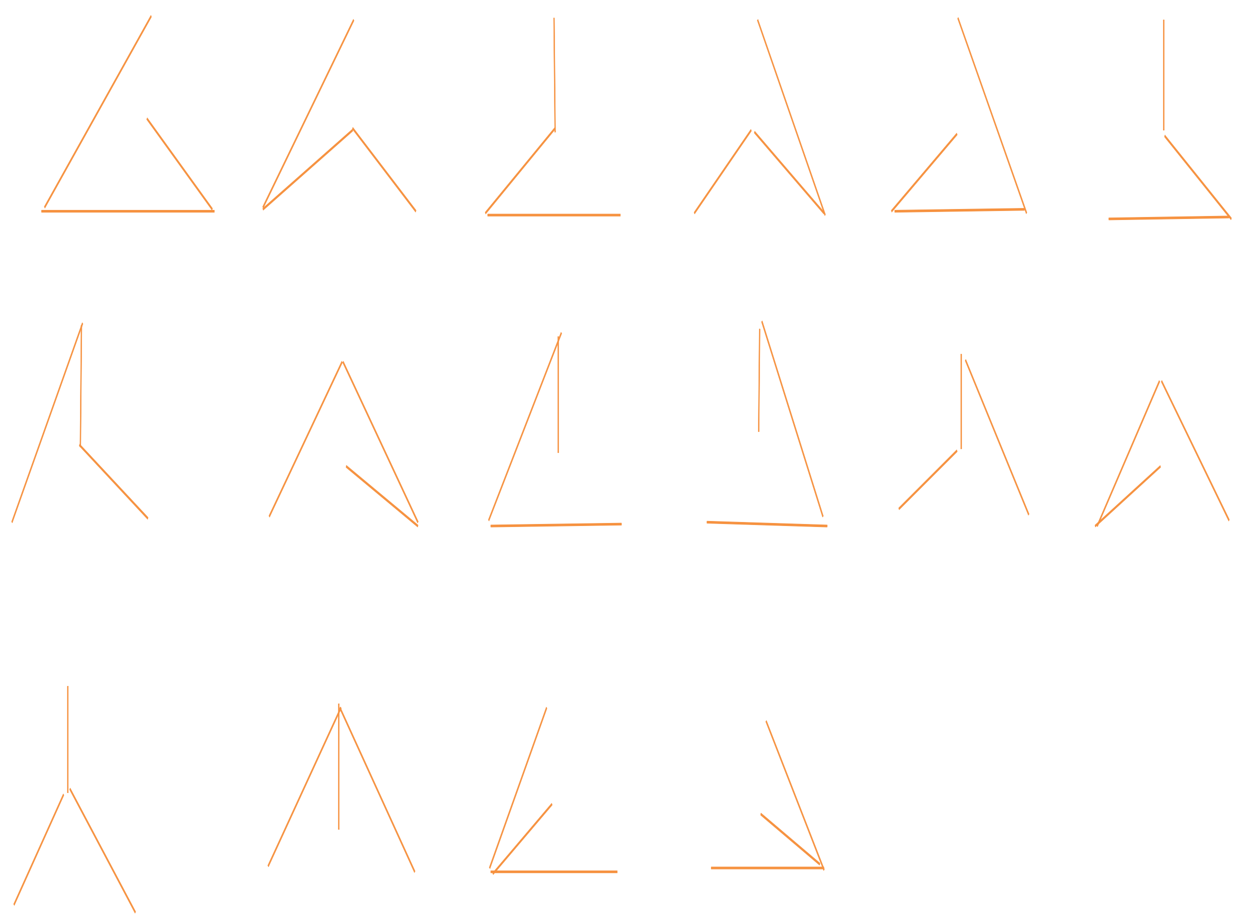
| |
|---|
| 0 |
| 4 |
| 3 |
| 8 |
| 10 |
| 19 |
| 28 |
| 48 |
| 75 |
| 124 |
| 198 |
| 323 |
| 520 |
| 844 |
| 1363 |

$NST_i =$

| |
|---|
| 1 |
| 5 |
| 16 |
| 45 |
| 121 |
| 320 |
| 841 |
| 2205 |
| 5776 |
| 15125 |
| 39601 |
| 103680 |
| 271441 |
| 710645 |
| 1860496 |

The complete spanning trees for the wheel n=3 (n+1=4) $P21_3$-3 = 16



For n = 12 the number of labeled spanning trees for the wheel on 13 points is 103680!

Note that since the spanning trees are not cycles it is not possible to find orbits for the sequence NST.

The above formula is also true for Helm graphs (adjoining a pendent edge to each node of the cycle of the wheel graph

The number of graph cycles for the wheel graph is known ( OEIS A002061) and is given by as simple polynomial W= $n^2 - n+1$. As a note, the sequence length mod(n) of the Perrin sequence for type 1 primes was found to be $n^2-1$(Board #2)

For p= type 1 primes the number of graph cycles for the wheel graph can be given by

Sequence length mod(p)+2-p

Example p=5 W=24+2-5 =21.

Other perrin like sequences or higher order perrin sequences may likely represent MISs,cycles or spanning trees of various types of graphs. This is a new area of application for graphs used in networks and communication theory. The problem of finding MISs is an NP hard optimization problem and it becomes less likely to find an efficient algorithm for an MIS of a graph.

RT