CHALKBOARD #8 [α,β] Classes of Elliptic Functions

The elliptic equation:

$F(x) = x^3 - x - 1$

Is a special form called the Weierstrauss form of a 4^{th} order polynomial:

$G(x) = y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$

By transformation (see Erdeyli, **Higher Transcendental Functions**, vol 2 Bateman Manuscript Project (1953) pg 304)

The equation for F(x) can be written as:

$F(x,y) = Y^2 = x^3 - (g_2/4)x - (g_3/4)$

We have assumed that the roots of the Perrin equations are roots to F(x,y) =0.
In this form we find that the Perrin elliptic equation is of the form where

$(g_2/4) = 1$ and $(g_3/4) = 1$

And

$g_2 = a_0*a_4 + 3*a_3^2 - 4*a_1*a_3$

$g_3 = 2a_1a_2a_3 + a_0a_2a_4 - a_0a_3^3 - a_1^2a_4 - a_2^3$

I define the general Perrin equation as (α,β) elliptic functions where,

$(g_2/4) = \alpha$ and $(g_3/4) = \beta$

Consider the [α,β] elliptic equation [4,5]

First we can solve for the roots as in Chalkboard #1

$$a := \left[\frac{-\beta + \left(\beta^2 + \frac{4\cdot\alpha^3}{27}\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{3}}$$

$a = 1.647$

$$b := \left[\frac{-\beta - \left(\beta^2 + \frac{4\cdot\alpha^3}{27}\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{3}}$$

$b = 0.809$

$r1 := a + b$

$r2 := a\cdot\exp\left(2\frac{\pi\cdot i}{3}\right) + b\cdot\exp\left(4\frac{\pi\cdot i}{3}\right)$

$\varepsilon := \exp\left(\frac{2\cdot\pi\cdot i}{3}\right)$

$r3 := a\cdot\exp\left(4\cdot\frac{\pi\cdot i}{3}\right) + b\cdot\exp\left(2\cdot\frac{\pi\cdot i}{3}\right)$

$r145 = 2.457$

$r245 = -1.228 + 0.726i$

$r345 = -1.228 - 0.726i$

The Perrin sequence for [4,5] can be calculated from the roots by taking the nth power
The 5th power is shown below

$n := 5$

$$S145 := r145^n + r245^n + r345^n$$

$S145 = 100$ with the resulting sequence:

3,0,8,15,32,100,203,560,..

where an=5*a(n-3)+4+a(n-2)

560=5*32+4*100

We can also calculate the negative sequence for (4,5)

$$SN145 := r145^{-n} + r245^{-n} + r345^{-n}$$

$SN145 = 0.31232$

Unlike the Perrin sequence [1,1] other negative sequences are not integers. It can be shown that the negative sequence is a series of rational fractions.

The numerator of the fraction is calculated from roots by:

$$(r145\,r245)^n + (r145\,r345)^n + (r245\,r345)^n = 976$$

The denominator is the ratio

$$\frac{976}{SN145} = 3125$$

resulting in the following negative sequence

$$3, \frac{-4}{5}, \frac{16}{25}, \frac{11}{125}, \frac{-144}{625}, \frac{976}{3125}, \frac{-3629}{15625}$$

As in the Perrin sequence the nth term of SN145 is calculated from the (n-1) and (n-3) terms

NUMERATOR: num(n-3)*beta^2 +Num(n-1)*alpha
DENOMINATOR: beta*Den(n-1)

The matrix representation of the [4,5] Perrin equation is

$$A45 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 4 & 0 \end{pmatrix}$$

$$v45 := \begin{pmatrix} 3 \\ 0 \\ 8 \end{pmatrix} \qquad A45 \cdot v45 = \begin{pmatrix} 0 \\ 8 \\ 15 \end{pmatrix}$$

With the identity matrix the series of isolated fixed points for sequence length n, Per45(n) is

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left| A45^{n} - I \right| = 2248$$

Sequence Per45(n) = 0,8,16,128,800,2248,.19456, 67768.

Note: mod(2) has a sequence length of 1 since $2^3 = 8$

1,1,1,1,1,1,1, ...mod(2)

mod(16) has a sequence length of n = 3
(8,0,8),(0,8,8),(8,8,0), (8,0,8)

As in chalkboard #6 we found the equivalence class (d,n) defined by the perrin sequence and negative perrin sequence

n = S1(d) - SN1(d)

since SN145 is defined by a rational fraction the actual relation for the equivalence class is:

n = S145(d) - SN145(numerator of d) + SN145(denominator of d) -1

Note that when the denominator is 1 the equation reduces to the [1,1]perrin sequence case. This equation is true for all [α,β] sequences.

Example (d,n) = (4,400) n = 32 - (144) + 625 -1 = 800

The discriminant and invariant properties.

For the equation x3 -g2/4x-g3/4 = 0

$$g2 := -2\cdot\left(r145^2 + r245^2 + r345^2\right)$$

$$\frac{g2}{4} = -4$$

$$g3 := -4\cdot r145\cdot r245\cdot r345$$

$$\frac{g3}{4} = -5$$

The discriminant is

$$\Delta := 16\cdot(r245 - r345)^2\cdot(r345 - r145)^2\cdot(r145 - r245)^2$$

$$\Delta = -6704$$

The alternative definition is

$$-16\cdot\left[4\cdot\left(\frac{g2}{4}\right)^3 + 27\cdot\left(\frac{g3}{4}\right)^2\right] = -6704$$

The algebraic modular j-invariant is defined as:

$$J := \frac{\left[-24\cdot 2\cdot\left(\frac{g2}{4}\right)\right]^3}{\Delta} \qquad J = -1055.77088$$

The j-invariant allows a parameterization of the various [α,β] elliptic curves. Particularly, it defines the isomorphism classes of elliptic curves. The above definition opens an infinite variety of Perrin sequences. This chalkboard provides a means of analyzing the periodicity of these sequences.

Using Numerical Transform Functions to Find Sequence Lengths

Mathcad provides both fast Fourier transforms (FFTs), which make use of vectors of length 2^m to speed up the calculation, and more generic discrete Fourier transforms (DFTs), which calculate transforms for signals of arbitrary length. The lowercase functions normalize by $\frac{1}{\sqrt{N0}}$ while the uppercase transform functions normalize by $\frac{1}{N0}$. In addition, theFFT functions **fft()** and **FFT()** accept only real-valued arguments. If a signal is entirely real, its Fourier transform will be complex-conjugate symmetric, so it is only necessary to calculate half of it, further speeding calculation.

Suppose you have a signal: $v_{k+3} := \mathrm{mod}\left(5 \cdot v_k + 4 \cdot v_{k+1}, pr\right)$

$$va_{k+3} := 100 v_k + 10 \cdot v_{k+1} + v_{k+2}$$

va transforms the data to a 3 vector e.g. (308)

Create a sampled data vector of length 2^m:

$$n_k := \frac{k}{f_s}$$

where $f_s = 1$ samples per second is the sampling frequency, as visible on the plot.

$v11_k := va_k$

$pr \equiv 13$

$Spec := fft(v11)$ is defined as

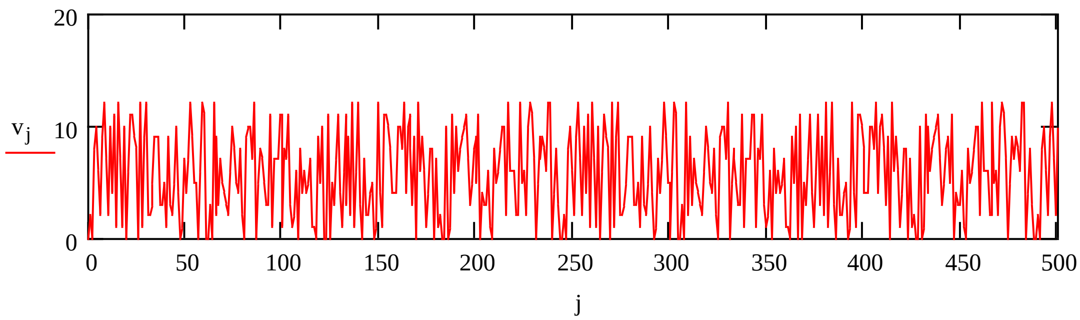
Find sequence length mod (pr) for the [4,5] Perrin sequence in this Case pr = 13

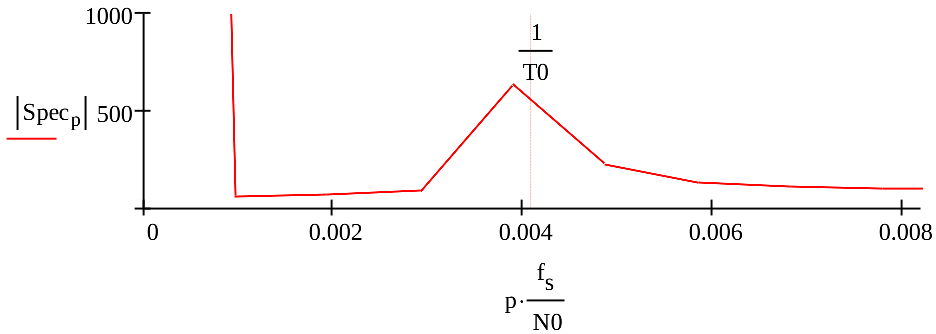
$$p := 0 .. \frac{N0}{2}$$

$N0 = 1024$ $\text{length}(Spec) = 513$

$Spec2 := FFT(v11)$ is the same, but normalized by 1/N0.

UNLIKE THE [1,1] PERRIN SEQUENCE NO OBVIOUS PATTERN FOR S145 (mod13) is observed. Signal processing finds a sequence of length 244





$N0 \equiv 2^{10}$

$T0 \equiv 244$

T0 = 244 is the period

$\frac{1}{T0} = 4.098 \times 10^{-3}$ signal frequency

$f_s \equiv 1$ sampling frequency

Other Sequence lengths found for the [4,5] Elliptic equation

Prime	Sequence Length	Prime	Sequence Length
2	3	13	244
3	26	17	144
5	None since 5|β	19	360
7	342	23	12166
11	120	29	12194

Note that in some cases the sequence length is p^2 -1 or p^3 -1.

Next Chalkboard- Generating Functions and Zeta Functions

RT