Chalkboard #9 The Generating Functions

In the past 8 chalkboards the discussion has been focused on integer and rational number sequences of numbers. In this board I will discuss the transformation of integer sequences into real number functions.

I derive a number of functions in x, F(x) with the following property.

$F(x) = f0 + f1*x + f2*x^2 + f3*x^3 ......+ fn*x^n$

where F(x) is a power series in the variable x and f0,f1,f2,...fn are integer or rational numbers that are defined by sequence relations f3=a*f0 + b*f1...
fn= a*f(n-3) + b*f(n-2) or fn= a*f(n-3) +/- b*f(n-1).

I will not show the details of the derivation of these generating functions, only results. Several papers are available on-line which discuss finding generating functions for sequences.
(See MIT course notes, week 11, Mathematics for Computer Science (A.R. Meyer and R. Rubenfeld, 2005)

1. Generating function for the Perrin Sequence

The first 18 terms of the power series are:
(for convenience I have split into two continuous series where B=B(x) and C=C(x))

$$B := 3 + 0\cdot x + 2\cdot x^2 + 3\cdot x^3 + 2\cdot x^4 + 5\cdot x^5 + 5\cdot x^6 + 7\cdot x^7 + 10\cdot x^8$$

$$C := 12\cdot x^9 + 17\cdot x^{10} + 22\cdot x^{11} + 29\cdot x^{12} + 39\cdot x^{13} + 51\cdot x^{14} + 68\cdot x^{15} + 90\cdot x^{16} + 119\cdot x^{17} + 158\cdot x^{18}$$

The generating function is A=A(x)

$$A := \frac{3 - x^2}{1 - x^2 - x^3}$$

A numerical solution with x = 0.2 shows that A = B+C

$$A = 3.1092436975 \qquad B + C = 3.1092436975$$

$$A - (B + C) = 0$$

2. Generating Function for the Inverse Perrin Sequence

$$BR := -3 + x - x^{2} - 2\cdot x^{3} + 3\cdot x^{4} - 4\cdot x^{5} + 2\cdot x^{6} + x^{7} - 5\cdot x^{8} + 7\cdot x^{9}$$

$$CR := -6\cdot x^{10} + x^{11} + 6\cdot x^{12} - 12\cdot x^{13} + 13\cdot x^{14} - 7\cdot x^{15} - 5\cdot x^{16} + 18\cdot x^{17} - 25\cdot x^{18}$$

The generating function is AR=AR(x)

$$AR := \frac{2\cdot x + 3}{x^{3} - x - 1}$$

$AR = -2.8523489933$ $\qquad$ $BR + CR = -2.8523489933$

$AR - (BR + CR) = 0$

As shown in an earlier board, the sum of the above generating functions A(x) and AR(x) is the isolated fixed point period sequence of the 3 torus Per11(x). [OEIS - A001945}

Adding terms of B+C and BR+CR give the sequence 0,1,1,1,5,1,7,8,5,19,11,23,35,27,64,which is A001945:

Below is the value of the sums at x = 0.2

$A + AR = 0.2568947042$

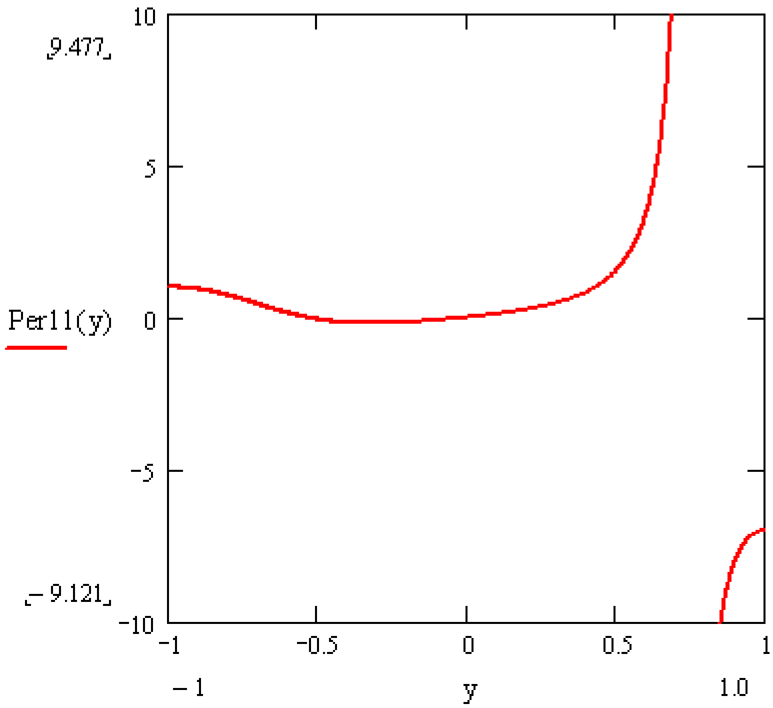
$(B + C) + (BR + CR) = 0.2568947042$

Combining the generating functions A(x) and AR(x) gives Per11(x)

$$Per11 := \frac{(-1)\cdot x\cdot\left(1+2\cdot x+x^2+2\cdot x^3+x^4\right)}{\left(x^3-x-1\right)\cdot\left(1-x^2-x^3\right)}$$

$$Per11 = 0.2568947042$$

Plot of Per11(y) (vertical axis) for -1.0< y <1.0 (horizontal axis) [y is used as the independent variable]



The radius of convergence r for the power series in x can be estimated from the limit of the ratio:

$$r = \lim_{n \to inf} \left| \frac{fn}{f(n+1)} \right|$$ This limit is about r = 0.75 where the value of Per11(x)

approaches infinity. For x<0 there are no zero roots so the function extends to - infinity.

Generating Functions: A general case.

[α,β] elliptic equations and Perαβ(x).

In board 8, I derived a general equation for calculating the number of isolated fixed points for the Perrin sequence and the Inverse Perrin sequence where in the later case the numbers can be rational fractions.

Perαβ = S1αβ - SNαβ(num) + SNαβ(den) - 1

Below are the necessary generating functions to derive the power sequence in x for Perαβ(x)

$$S1\alpha\beta := \frac{3 - \alpha \cdot x^2}{1 - \alpha \cdot x^2 - \beta \cdot x^3}$$

$$SN\alpha\beta num := \frac{f0 + (f0 \cdot \alpha + f1) \cdot x + (f2 + f1 \cdot \alpha) \cdot x^2}{1 + \alpha \cdot x - \beta \cdot x^3}$$

$$SN\alpha\beta den := \frac{1}{1 - \beta \cdot x}$$

In the sum, the generating function of the number 1 must also be calculated:

ONE= 1(x) = 1+1*x+1*x^2 +......1*x^n

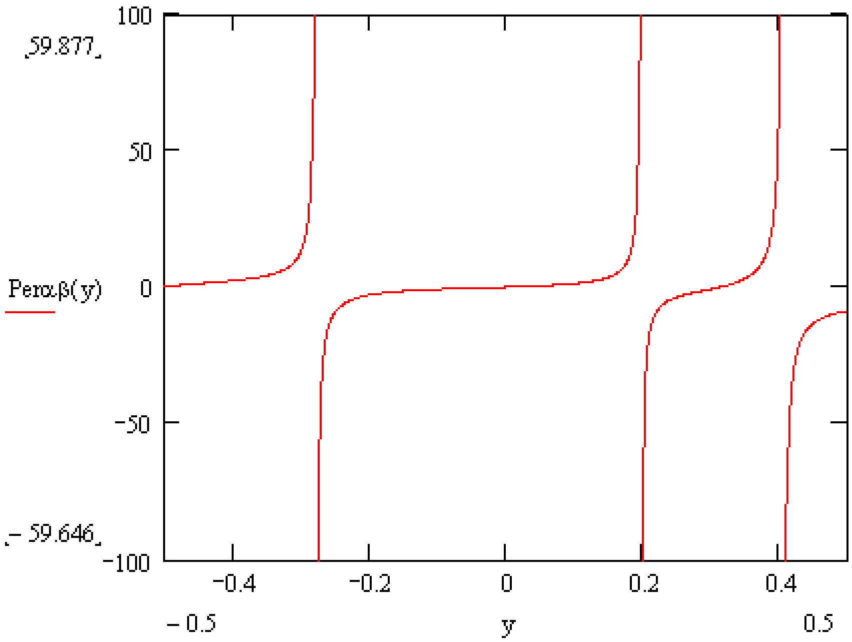
SNαβden is the general case of this series when β does not equal 1.

1+ βx+ (βx) + $(\beta x)^2$ + $(\beta x)^3$ +....$(\beta x)^n$

$$ONE := \frac{1}{1 - x}$$

$$Per\alpha\beta := S1\alpha\beta - SN\alpha\beta num + SN\alpha\beta den - ONE$$

Plot of Perαβ(y) (vertical axis) for -0.5< y < 0.5 (horizontal axis) where α=4 and β=5



The radius of convergence r for the power series in x for Per45(x) can be estimated from the limit of the ratio:

$$r = \lim_{n \to inf} \left| \frac{fn}{f(n+1)} \right|$$

This limit is about r= 0.20 where Per45(x) approaches infinity.

The denominator (1+4x-5x3) has zeros for x<-0.2 so becomes infinite.

Invariant Properties

Previously I defined the elliptic equation in terms of Weierstraus invariants g2 and g3 where α=g2/4 and β=g3/4

If e1, e2, and e3 are roots of x^3 -(g2/4)x - (g3/4) then the invariant properties are:

$$g2 = -4*(e1*e2+e1*e3+e2*e3) = -4*\sum_{(i,j)} ei*ej = 2*(e1^2+e2^2+e3^2)$$

$$g3 = 4*e1*e2*e3 = 4*\prod_{i} ei$$

*Also since 1/e1 +1/e2 +1/e3 = e1*e2+e1*e3+e2*e3/e1e2e3 = -(g2/4)/(g3/4) = -α/β*

The first three terms f0, f1, and f2 are defined for the sequences Sαβ and SNαβf

For Sαβ

f0 = 3

f1 = 0

f2 = g2/2 = 4α/2

For SNαβnum

f0num = 3

f1num = $\sum_{(i,j)} (ei \cdot ej)$

f2num = $f1num^2$

For SNαβden

f0den = 1

f1den = $\prod_{i} ei$

f2den = $f1den^2$

In the next board I will discuss other functions for Perαβ(x) which are generating functions for the orbit counts and number of isolated fixed points of the representative matrices of the 3-torus. These are called the dynamical zeta functions which are important functions for integer sequences and topology.

RT