Calculation of Perrin Numbers using a modified Beta function and an Application to Multiple Zeta Values

The following references to the subject of this Chalkboard can be found in:

(1) D.J.Broadhurst and D. Kreimer arXiv/9609128v3 18 Nov 1996
Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops

(2) M. Waldschmidt, MZV IMSc 2011, Lectures on Multiple Zeta Values , Updated: 16 June 2015

(3) J. Blumlein, D.J. Broadhurst, J.A.M. Vermaseren, arXiv/0907.2557v2 14 Nov 2009
The Multiple Zeta Value Data Mine

In Chapter 15, I introduced a modified Incomplete Beta function for calculating the Nth term of the Perrin sequence when N is prime.

Using the Incomplete Beta Function Bz(m,n)
with z = 1

$$Bz(m,n) := \int_0^z t^{m-1}\cdot(1-t)^{n-1}\,dt$$

$$MBz(m,n) := [(m+n)\cdot(m+n+1)\cdot Bz(m+1,n+1)]^{-1}$$

This equation correctly calculates Perrin(N) from the sigma orbit, orb1σ(N) for primes where:

$$\text{orb1}\sigma(N) = 1/N^* \sum_{\left(\frac{N}{d}\right)} \left(P(d)\cdot\mu\left(\frac{N}{d}\right)\right)$$

$$\text{Perrin}(N) = N^* \sum_i MBz(m,n)_i$$

where i is 0...floor((n-3) /6)

m= (n-3)/2 -3i and n= 1+2i

For prime numbers n is an odd integer 1,3,5,.... and m can be even or odd.

If we want to calculate the Perrin(N) for even numbers the same equations can be used with a slight modification.

$$\text{Perrin}(N) = N^* \sum_i MBz(m,n)_i +2$$

where i is 0...floor((n-6) /6)

m= (n-6)/2 -3i and n= 2+2i

Example

N=36

(N-6)/2 =15   i = 0...floor(30/6)

m = 15- 3i   n = 2+2i

MBz(15,2)= 8;
MBz(12,4) =113.75
MBz(9,6)=333.667
MBz (6,8)=214.5
MBz(3,10)=22,
MBz(0,12)=0.083

SUM = 692.0:

Perrin(36) = 36*692 +2 = 24914

It can be shown through other examples that for all odd numbers N,   Perrin(N) is calculated from the same equations used for primes using N* $\sum_{i} MBz(m,n)_{i}$

where n = 1+ 2i and all even numbers N using N* $\sum_{i} MBz(m,n)_{i}$ **+ 2 where n = 2+2i.**

Calculating the Number of Irreducible multiple zeta values at weight n

The sigma orbit is only a divisor of Perrin(N) when N is prime.

For composite numbers d|N so the full mobius transform μ of orb1σ must be used.

orb1σ(N) = 1/N* $\sum_{\left(\frac{N}{d}\right)} \left(P(d) \cdot \mu\left(\frac{N}{d}\right)\right)$

An Example for N = 36 is shown below.   These values are also shown in the table at OEIS A113788. 0,1,1,0,1,0,1,1,1,1,2,2,3,3,4,5........ (a(36) = 687)

Continuing with the example above the mobius numbers μ(d), are used at each divisor of 36

d1 = 1 $\mu(1)$= 1   36 *1

MBz(15,2)= 8;
MBz(12,4) =113.75
MBz(9,6)=333.66667
MBz (6,8)=214.5
MBz(3,10)=22,
MBz(0,12)=0.08333
SUM = 692

d2=2 $\mu(2)$=-1,   18*2

(18-6)/2 = 6

MBz(6,2)= 3.5
MBz (3,4)=5
MBz(0,6) = 0.16667
SUM = 8.6667

d3=12 $\mu(3)$=-1 12*3
(12-6)/2=3

MBz(3,2)=2
MBz(0,4)=0.25
SUM = 2.25

d4=9 $\mu(4)$=0   9*4
(9-3)/2 = 3   This calculation not required since μ(4) = 0

d6=6 $\mu(6)$=1
(6-6)/2 = 0

MBz(0,2) = 0.5
SUM = 0.5

d9=4 d18=2 and d36 = 1   Since all $\mu(9)$ = 0 and μ(18) = μ(36) = 0
no calculations are required

orb1σ = 692-(1/2)*8.667-(1/3)*2.25+(1/6)*0.5 = 687

This value agrees with the OEIS value.

An Euler sum is defined as alternating signs of a multiple zeta value

$$\zeta\left(s_1 \ldots\ldots s_k, \sigma_1 \ldots\ldots \sigma_k\right) \;=\; \sum_{(nj>nj+1>0)} \prod_{j=1}^{k} \frac{\left(\sigma_j\right)^{n_j}}{\left(n_j\right)^{s_j}}$$

with σ_j =+/- 1 and s_j are positive integers. The weight of the Euler sum is

$\sum_j s_j$ and depth is k.

In Chapter 11 I showed that the sigma orbit

orb1σ(n) = 1/n* $\sum_{dn}\left(P(d)\cdot\mu\left(\frac{n}{d}\right)\right)$

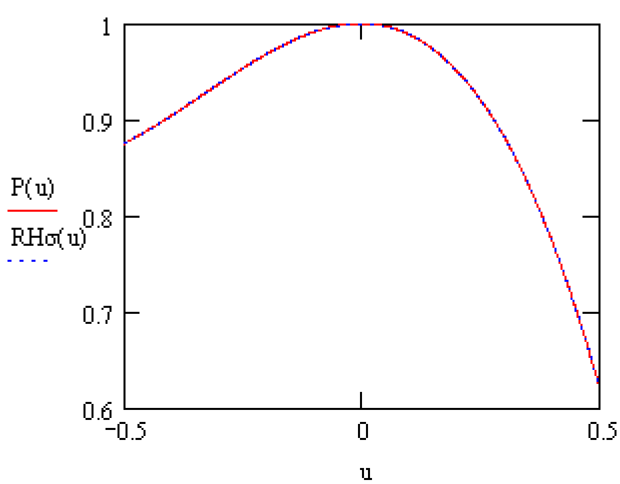
is related to the Perrin generating function

$$RH\sigma(u) := \left[\prod_j \left(1-u^j\right)^{orb1\sigma_j}\right]$$

$$P(u) := 1 - u^2 - u^3$$

$$P(u) := RH\sigma(u)$$

The graph of both functions is shown below



David Broadhurst (1,3) has used these equations to find that the the number of irreducible Euler sums of weight N and depth k can be obtained from the mobius transform of the Perrin numbers which divide N. Table 17 in (3) shows the number of basis elements for Multiple Zeta Values as a function of weight and depth when the MVZs are expressed as Euler sums in the minimal depth representations.

In this section, I will demonstrate how these basis elements at each depth can be calculated from the modified Beta function.

Example using the reults from weight N = 36

The basis for a Q-vector space spanned by the MVZ's is an algebra giving a linear combination of independent numbers.(2). For the Euler sums these numbers can be obtained from the modified Beta function as multiples of (m,n).

It is anticipated that the number of irreducible basis elements is an integer at each depth for a given weight. The maximun number of levels for the depth are given by floor((N-6)/6) +1 for even weights and floor((N-3)/6) +1 for odd weights.

The following values for MBz(m,n)*μ(d) are grouped by multiples of (m,n)

MBZ(15,2) = 8 depth 2

MBZ(12,4) - 1/2*MBz(6,2) = 112 depth 4

MBZ(9,6) - 1/3*MBz(3,2) = 333 depth 6

MBZ(6,8) - 1/2*MBz(3,4) = 212 depth 8

MBz(3,10) = 22 depth 10

MBz(0,12) - 1/2*MBz(0,6) -1/3*MBz(0,4) +1/6*MBz(0,2) = 0 depth 12

The sumation at each depth = 8 + 112 + 333 + 212 + 22 + 0 = 687.

The "Euler Triangle of irreducibles" (1) shows the number of irreducible basis at several depths. The vertical sequence of values can be found the following OEIS tables:

Depth	OEIS
3	A001840
4	A006918
5	A011795
6	A011796
7	A011797
8	A031164
9	A245559

It is also found that these values can be calculated from the modified incomplete beta function. The m^{th} vertical number under depth n can be found when m = (N-6)/2 for n even and m = (N-3)/2 for n an odd number. Then:

Entry (m,n) = MBz(m,n) + $1/d_1$*MBz(m/d_1,n/d_1)*$\mu(d_1)$ + $1/d_2$*MBz(m/d_2,n/d_2)*$\mu(d_2)$ +…

Where entry(m,n) is the m^{th} non zero number in column n of Table I in reference (1) and d_i are divisors of m and n [e.g d_i ={2,3,4,5,6…..}] and $\mu(d_i)$ is the mobius function at d_i.

Example: Entry (10,5) = MBz(10,5) + 1/5 * MBZ(2,1)* $\mu(5)$ = 200.2 +1/5(1)*(-1) = 200.

Entry (6,8) = MBz(6,8) + ½ * MBZ(3,4)* $\mu(2)$ = 214.5 + ½ * (5) = 212.

The application of these numbers to knot theory and Feynman diagrams is not the purpose of this chapter and the interested reader should consult the above mentioned references.

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