Perrin's First Sequence and Other Isomorphic Recurrence Sequences

"*The surprising proposition from Chinese origin which is the subject of question 1401 would produce, if it is true, a criterion which would be more efficient than the Wilson Theorem to check whether a given number m is prime or not: it would be suffice to compute the residues modulo m of the successive terms of the recurrence sequence*

$$u_n = 3u_{n-1} - 2u_{n-2}$$

with the initial conditions $u_0 = -1$ *and* $u_1 = 0$."(*)

* R. Perrin, L'Intermediaire des mathematiciens, Query 1484, v6, 76-77 (1899)
Translation to English provided by Dr. Michel Waldschmidt.

In this chapter, I would like to compare various recurrence sequences that are related to the Perrin and Fibonacci sequences. Perrin mentions the above sequence as a preface to his own named sequence in 1899. He is referring to two methods known at that time of finding primes: Wilson's Theorem (1770) and the Chinese Remainder Theorem. Wilson proposed the sequence (N-1)! + 1 and found that when N is prime,

$$(N-1)! + 1 = 0 \bmod N$$

Since computing (N-1)! when N is large is difficult, Perrin was seeking alternative algorithms.
Wilson's Theorem can be used to show that if p is a prime such that p = 1 mod 4 then if
p = 2*m +1 with m = 2k, then;

$$(m!)^2 = (-1) \bmod p$$

where (-1) is a quadratic residue or m! is a square root of (-1) mod p.

The Chinese remainder theorem can be also used to find prime numbers and is applied to Perrin's first equation above.

Let $N = a \bmod r$ and $N = b \bmod s$
Then $N = m \ast r + a$ and $N = m' \ast s + b$

where m and m' are integers and r, s are relatively prime: (r, s) = 1

Let $U \ast r + V \ast s = 1$ with u and v being integers. Then,
$U \ast r \ast N + V \ast s \ast N = N$
$U \ast r \ast (m' \ast s + b) + V \ast s \ast (m \ast r + a) = N$
$r \ast s \ast (U \ast m' + V \ast m) + (U \ast r \ast b + V \ast s \ast a) = N$

This implies $N = (U \ast r \ast b + V \ast s \ast a) \bmod (r \ast s)$

Example

$N = 17 = 5 \times 3 + 2 = 7 \times 2 + 3$ and $(3, 2) = 1$

Then let U = 1 and V = -1 so $U \times r + V \times s = 1 \times 3 + (-1) \times 2 = 1$

This implies $N = (1 \times 3 \times 3 + (-1) \times 2 \times 2) \bmod (3 \times 2)$

Or

$17 = 5 \bmod 6$

It must have been known at the time of Perrin that any integer can be formed from the sum of products of 2 and 3.
Associating U = 1 with 3 and V = -1 with 2 Perrin's first recurrence sequence is

(1) $u_n = (3 \times U)\, u_{n-1} + (2 \times V) u_{n-2}$

This sequence would have an associated quadratic function;

$x^2 - 3Ux - 2V = 0$

with solutions x = 1 and x = 2. The initial numbers of this sequence would be calculated as;

$u_0 = 1^0 + (2)^0 = 2$ and $u_1 = 1^1 + (2)^1 = 3$

Perrin was seeking the condition that when n is prime;

(2) $u_n = 0 \bmod n$.

Using these initial conditions, $u_0 = 2$ and $u_1 = 3$ sequence (1) does not satisfy condition (2).

However condition (2) is satisfied when u_0 and u_1 are modified by subtracting three, then $u_0 = -1$ and $u_1 = 0$ in sequence (1). Condition 2 is also satisfied for any set of numbers such that (r, s) = 1 and $U \times r + V \times s = 1$ with U > 0. In general the sequence is said to be *realizable* as discussed in Chalkboard #6.

When u_0 and u_1 are modified by subtracting two the sequence becomes a linear recurrence sequence of Mercenne numbers. A Mercenne number is $M_n = 2^n - 1$. It is likely that Perrin was familiar with these numbers. The Mercenne prime is any number M_n that is prime. A list is given in OEIS A000668.

As in the case of the Wilson Theorem these sequences become large for modest numbers n.

For realizable sequences there is a mobius transform of the sequence and a corresponding sigma orbit giving the number of unlabeled maximal independent sets on the n-cycle graph. As in the case of the Perrin sequence, one might expect Perrin's first sequence or equation (3) to have a combinatory function similar to the equation derived in Chapters 14 and 15.

For prime numbers N, the corresponding sigma orbit for u_n in equation (3) can be found from the modified beta function as follows:
Let m = (N-2) -3i and n = 1 + 3i where i = 0……floor((N-2)/3))

Then

(4a) $\sigma 1(N) = \Sigma_i \, MBz(m_i,n_i)$

(4b) $orb1\sigma(N) = [\sigma 1(N) \ast (N-1) \ast 6]/N$

(4c) $u_n = N * orb1\sigma(N)$

(4d) $M_n = u_n + 1$

Example: N = 23 u_n = 8388606 M_n = 8388607

$\sigma 1(N)$ = MBz(21,1) + MBz(18,4) + MBz(15,7) + MBz(12,10) + MBz(9,13) + MBz(6,16) + MBz(3,19) + MBz(0,22)

= 1 + 332.5 + 7752 +29393 +22610 + 3391.5 + 70 + 0.04545..

= 63550.04545.

.

$orb1\sigma(N) = [\sigma 1(N) \ast (23-1) \ast 6]/23 = 63550.04545.. \ast 22 \ast 6 /23 = 364{,}722$

$u_n = N * orb1\sigma(N) = 23 * 364{,}722 = 8388606$

From (4d) the Mercenne number at N(prime) is determined (N>3). (see OEIS A001348)

The calculation of u_n for N not prime follows the following conditions:

N is odd and not prime	$u_n = 6 \ast [\sigma 1(N) \ast (N-1) - 1]$
N is even	$u_n = \sigma 1(N) \ast (N-1) \ast 6 + 2$
N is even and $N\|2^i$ i=1,2,3…	$u_n = \sigma 1(N) \ast (N-1) \ast 6 - 4$

Perrin continues: "*I have met another recurrence sequence which seems to have the same property; this is the sequence of general form,*

(3) $v_n = u_{n-2} + u_{n-3}$

with initial values $v_0 = 3, v_1 = 0, v_2 = 2$."

"*It is easy to prove that* v_n *is divisible by n, if n is a prime; I have checked that this does not happen otherwise, up to rather large values of n; but it would be interesting to know what happens in reality, especially since the sequence* v_n *produces numbers which grow much slower than the sequence* u_n *(for n = 17 for instance, we find* u_n*= 131070 while* v_n*= 119), and allows some simplifications for the computation when n is a large number.*"

It is noted that with the original conditions for (1); $u_0= 2$ and $u_1= 3$, then at n=17

u_n= 3 mod 6 and u_n= 3 mod 17

whereas, for $u_0 = -1$ and $u_1= 0$

u_n= 0 mod 6 and u_n= 0 mod 17

In contrast, for Perrin's equation (3),

v_n= 5 mod 6 and v_n= 0 mod 17

with (3,2) = 1.

The sixth Mercenne prime is 131071 where $M_n - u_n = 1$. There are then 10 Mercenne numbers which are *not* prime for 1 < n < 17. (As of today the largest known Mercenne prime is $M_n = 2^{74,207,281} -1$) for which a general form is n = 6m +1 (m= 12,367,880). It can easily be shown that a *perfect number* defined by $2^{p-1}*(2^p -1)$ is of the form:

$$2^{p-1}*(2^p -1) = [S^2 +3*S +2]/2$$

Where S = σ1(p) *(p-1) * 6 with σ1(p) defined in equation (4a) above!

In this chapter I will compare various initial values v_0= 3 + k_1, v_1= 0, v_2= 4α/2 + k_2
I will examine equation (3) using other initial values. In Chapters 8 and 9, I discuss [α,β] Perrin sequences where x^3-αx-β= 0 is the associated elliptic equation. Here [1, 1] Perrin is Perrin's original sequence with k_1= k_2= 0 (α= 1).

In Chapter 11, I discuss the (2, 1) Perrin sequence with initial values v_0= 3, v_1= 0, v_2= 4.
It is interesting that the associated elliptic equation to this sequence has solutions: ϕ, 1/ϕ, and (-1) where ϕ is the golden ratio (ϕ= 1.618033988....) and the limit as n increases is

$$v_n/v_{n-1} = \phi$$

This contrasts with the limit as n increases for Perrin's sequence:

$$v_n/ v_{n-1} = \rho$$

where ρ is the plastic number (ρ= 1.324717957...).

In the 1960's, G. Andrews [1] developed a formula to calculate the nth term of a Fibonacci sequence (1,0,1,1,2,3,5,8,13,…).

Define m(i) = ((n-2)-5*i)/2) where i is a summation index from minus to plus infinity. For application purposes it is best to define i= 0,-1,1,-2,2,-3,3,….where i(max) is such that m(i) ≥ 0.

(5) $F(n) = \sum_i (n-2)!/(\text{floor}(m(i))!*(n-2 - \text{floor}(m(i)))!)* (-1)^i = \sum_i C(n-2, \text{floor}(m))*(-1)^i$

where C(n,m) is the binomial coefficient.

Example: n = 15

i	m(i)	Floor(m(i))	$(-1)^i$	ith Term of (5)
0	6.5	6	1	1716
-1	9	9	-1	-715
1	4	4	-1	-715
-2	11.5	11	1	78
2	1.5	1	1	13

F(15) = SUM 377

Equation (5) can then be used to calculate the nth term of the (2,1)Perrin sequence, $P_{21}(n)$, and Lucas sequence mentioned above.

(6a) $P_{21}(n) = F(n) + F(n+2) + (-1)^n$

(6b) $L(n) = F(n) + F(n+2)$

In this section let the initial values of the Perrin sequence be $v_0 = 3 + k_1$, $v_1 = 0$, $v_2 = 4\alpha/2 + k_2$ where $k_1 = k_2 = 5$ and $\alpha = 1$. First consider two sequences P1(n) = 3,0,2,3,2,5,5,7,10.. and P2(n) = 8,0,7,8,7,15,15,22,30.. These two sequences are defined to be *isomorphic recurrence sequences*. If each term is calculated mod 5, then for each term P1(n) mod 5 – P2(n) mod 5 = 0. The Pisano type period mod 5 is 24. In general, the Pisano type period modulo r is equal for both sequences with the following condition:.

Conjecture 1: The Pisano type periods for two *isomorphic recurrence sequences* are equal for all mod r with $r \geq 1$ ***except*** for $r = |4\alpha^3+27\beta^2|$ the discriminant of the associated elliptic curve for $x^3+\alpha x+\beta$.

The discriminant and period of the associated elliptic equations are discussed in Chalkboard 8 where the discriminant of x^3-x-1 is -23.

The difference P2(n)-P1(n) can be calculated from OEIS A099559 with A(n):

(7) $A(n) = \sum_i C(n-4k, k+1)$

with k = 0….floor(n/5).

(8) P2(n) – P1(n) = A(n+1) – A(n-23)

Exercise show: P2(40) – P1(40) = A(41) – A(17).

Euler's Pentagonal Numbers.

This section is taken from D. Cohen [2].

A partition of an integer n is a k-tuple $(\lambda_1,\lambda_2,\ldots\lambda_k)$ with $\lambda_i > \lambda_{i+1}$ and $n = \lambda_1+\lambda_2++\lambda_k$ The number k is the length of the partition. For example the number 12 can be expressed in various lengths (6,3,2,1),(7,5), (12), (3,3,2,2,2) with respective lengths 4,2,1,5 respectively.

Let $E_{x,y}(n)$ = number of partitions of n into an even number of distinct parts congruent to 0, $\pm$ x(mod y) and $O_{x,y}(n)$ = number of partitions of n into an odd number of distinct parts congruent to 0, $\pm$ x(mod y).

Euler considered the case where x = $\pm$1 and y = 3 since any integer can be expressed as a sum of 2's and 1's. For example n=7 can be expressed as (7), (4,3), (6,1), (5,2), and (4,2,1). Each partition consists of a set of numbers congruent to 0 or $\pm$ 1(mod 3). Also each of these have distinct parts without duplication (e.g. (3,2,2) is not a member of this set of numbers.

Euler found that the number of even and odd partitions into distinct parts is given by:

$$E_{13}(n) – O_{13}(n) = (-1)^k \text{ if } n = k*(3k\pm1)/2$$
$$= 0 \quad \text{otherwise}$$

The number $k*(3k\pm1)/2$ is called a generalized pentagonal number (OEIS A001318). The sequence is 0,1,2,5,7,12,15,22,26… For the partitions of 7 above, there are 3 even and 2 odd partitions so the difference is $3-2 = (-1)^2 = 1$, since for k = 2 n is a pentagonal number.

Euler's recursion formula for the number of partitions of a number into partitions of *all* lengths is

$$p(n) – p(n-1) – p(n-2) + p(n-5) \ldots p(n-\Pi) = 0 \quad \text{where } \Pi \text{ is a pentagonal number.}$$

(9) $$p(n) = \sum_{k\geq1} (p(n-\Pi(k))+(p(n-\Pi(-k))*(-1)^{k+1}$$

Example: p(18) = p(17) + p(16) - p(13) - p(11) + p(6) + p(3)
= 297 +231 – 101 – 56 + 11 + 3 = 385

(see OEIS A00041for this sequence of the number of partitions of n).

Rogers-Ramanujan Identities

It is the combinatorial interpretation of these identities that is the focus of this and the next chapter. A brief description is found in Weisstein [3]. From the discussion of pentagonal numbers we are seeking the number of partitions of the integer n into parts not congruent to 0 , $\pm$ x(mod y).

Theorem I [[2],Theorem 4] Let y not equal 2*x. If $A_{x,y}(n)$ is the number of partitions of n into parts **not** congruent to 0, $\pm$ x(mod y) then:

(10) $$A_{x,y}(n) = \sum_k p(n – k*[yk \pm (y-2x]/2)*(-1)^k$$

with k = 0 to infinity and where $k*[yk \pm (y-2x]/2$ is the "sieve".

If we look at y = 5, (mod 5) the first two Rogers-Ramanujan identities can be obtained.

(±x,y)	Sum	First few sieves k = 0..3	OEIS
(±1,5)	p(n-k*(5k±3)/2)	0,1,4,7,13,18,27	A003106
(±2,5)	p(n-k*(5k±1)/2)	0,2,3,9,11,21,24	A003114

The first identity is interesting since it calculates a series (A003106) giving the number of partitions of n into parts 5k +2 or 5k+3. The second series gives the number of partitions of n into parts 5k +1 or 5k+4.

Corollary to Theorem I. The number of partitions of n into parts 5k+2 and 5k+3 is given by the recurrence sequence p(n) = p(n-1) + p(n-4) – p(n-7) – p(n-13) + p(n-18)…

Show that p(18) = 15 from Theorem 1 and its corollary.

To conclude Perrin's 1899 Query: "*The same method of proof, applied to one of these sequences, will probably be applicable to the other, assuming that the property is true for both of them: it remains only to find it.*"

The connection of the Rogers-Ramanujan identity recurrence sequence and the isomorphic recurrence sequence of Perrin's sequence will be investigated in the next Chapter.

1. G.E. Andrews, *Fibonacci Quarterly* 7, (1969) 113-130.
2. D. I.A. Cohen, *Journal of Combinatorial Theory*, Series A 27, (1979) 325-332.
3. E.W. Weisstein, CRC Concise Encyclopedia of Mathematics, 2nd Ed. (2003) 2588-2589.

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