Appendix to 17

Some Observations on the Restricted Rogers Ramanujan Identities

Chalkboard 17 discussed the partition of an integer into parts (+/-)3 mod 5, particularly OEIS A003106. There are various depths or lengths k of the partitions. The sum of these depths is the number of partitions of an integer N. Let p(N,k) represent the restricted partition of an integer N into parts of depth k then:

A003106(N) = p(N,1) + p(N,2) + p(N,3) + ……. + p(N,k)

(e.g.) A003106(18) = 15 = 1+ 0 + 4 + 2 +2 +3 +1 + 1+ 1

The basis representation of each p(N,k) is composed of the words {2} and {3} which sum to a number equal to the modulus of N. The maximum value of k is equal to the ceiling(N/2).

We define the prime basis representation of a depth k as the minimum basis of depth k for an integer of a given modulus.

For example, the prime basis representation for k=7 and an integer 14 = 4 mod 5 is (2,2,2,2,2,2,2). The upper limit of this basis is the limit basis representation (3,3,3,3,3,3,3) or 21 = 1 mod 5.

There are 2 integers of value 4 mod 5 between the prime and limit basis; 14 = 4 mod 5 and 19 = 4 mod 5. This means that for numbers equal to 19 and any number 4 mod 5 greater than 19 we need two basis representations to describe the partition of N into 7 parts, i.e. (2,2,2,2,2,2,2) and (3,3,3,3,3,2,2).

The prime basis has m =(N-14)/5 'solutions' or objects. For N= 19 there is only one solution m = 1 so the basis can be expanded as (2 + 5*1,2,2,2,2,2,2) = (7,2,2,2,2,2,2). The second basis is exactly (3,3,3,3,3,2,2) so there are a total of 2 solutions for k = 7 and N = 19.

The number of ways of partitioning the number N having (N-14)/5 and (N-19)/5 objects is more difficult when finding the number of k =7 partitions for any number 4 mod 5 greater than 19.

N = 24 requires partitioning of 2 = 2+0+0+0+0+0+0 = 1+1+0+0+0+0+0 and 1 = 1+0+0+0+0+0+0 into 7 parts. Then for the prime basis (2+5*2,2,2,2,2,2,2) and (2+5*1,2+ 5*1,2,2,2,2,2) are the partitions (12,2,2,2,2,2,2) and (7,7,2,2,2,2,2).

The partitioning of the second basis representation of mixed elements consisting of 5 "3s" and 2 "2s" requires more computation. Since we can distinctly partition 1 =1+0+0+0+0+0+0 = 0+0+0+0+0+1+0 there are two solutions: (3+5*1,3,3,3,3,2,2) and (3,3,3,3,3,2+5*1,2) = (8,3,3,3,3,2,2) and (3,3,3,3,3,7,2)! The number 24 has a total of 4 partitions of length 7 in parts (+/-)3 mod 5.

The above analysis can be used with any integer to partition into parts (+/-)3 mod 5 of arbitrary lengths.

Choose N = 24 and k = 5. The only basis representation of a number equal to 4 mod 5 is (3,3,3,3,2) since the prime basis (2,2,2,2,2) = 0 mod 5 and the limit basis (3,3,3,3,3) = 0 mod 5. In this instance there can be only one basis of k = 5 between the prime basis and limit basis in this case.

Then m =(24-14)/5 = 2 so there are 4 **distinct** ways of partitioning the number 2 into the basis (3,3,3,3,2).

2 = (2,0,0,0,0) = (0,0,0,0,2) = (1,1,0,0,0) = (1,0,0,0,1).

The problem is similar to partitioning 2 objects into 4 red boxes and 1 yellow box.

To show that there can be more than 4 distinct partitions of 24 for a given k, consider k = 4.

Again, the prime basis representation for k = 4 is (2,2,2,2) = 8 =3 mod 5 and the limit basis is (3,3,3,3) =12= 2 mod 5 so there is only one basis for k = 4 and N = 4 mod 5 equal to (3,2,2,2). Then the solution (24-9)/5 = 3 so we have 3 object to put into 1 red and 3 yellow boxes. There are 7 distinct ways to do this:

(3,0,0,0), (0,3,0,0), (2,1,0,0), (1,2,0,0), (0,2,1,0), (0,1,1,1), (1,1,1,0) resulting in the 7 partitions of 24 into 4 parts (+/-) 3 mod 5.

(3+3*5,2,2,2), (3,2+3*5,2,2), (3+2*5,2+1*5,2,2), (3+1*5,2+2*5,2,2), (3,2+2*5,2+1*5,2),

(3,2+1*5, 2+1*5, 2+1*5), and (3+1*5, 2+1*5, 2+1*5,2) =

(18,2,2,2), (3,17,2,2), (13,7,2,2), (8,12,2,2), (3,12,7,2), (3,7,7,7) and (8,7,7,2).

Using the prime basis and limit basis for each depth k, the number of basis representations for each k and integer modulo 5 can be determined.

Table of Number of Basis Representations Required to Partition each integer mod 5 (horizontal) by depth k mod 5 (Vertical).

	N' mod 5	0	1	2	3	4
k mod 5						
0		(N/5) + 1	N/5	N/5	N/5	N/5
1		N/5	N/5	(N/5) + 1	(N/5) + 1	N/5
2		(N/5) + 1	(N/5) + 1	N/5	N/5	(N/5) + 1
3		N/5	(N/5) + 1	(N/5) + 1	(N/5) + 1	(N/5) + 1
4		(N/5) + 1	(N/5) + 1	(N/5) + 1	(N/5) + 1	(N/5) + 1

N is a number 0 mod5. (0,5,10,15…) with $N \leq k < (N+5)$. N' is an integer.

As an example, how many basis representations are needed to partition an integer 2 mod 5 of depth 2 mod 5? The Table coordinate (2,2) indicates there are N/5 basis needed. For each bases there are (N'-M)/5 objects to consider, where M = sum of each basis elements.

If we consider a basis of depth 7 = 2 mod 5 then N = 5 indicating only one basis representation is required for a number equal to 2 mod 5. The k = 7 basis representation is (3,3,3,2,2,2,2) where M = 3+3+3+2+2+2+2 = 17.

Then given N = 32 there are (32-17)/5 = 3 objects to consider for this basis. Equivalently, how many distinct ways are there to distribute 3 similar objects (e.g. tennis balls) into 3 red boxes and 4 yellow boxes allowing some boxes to remain empty? In combinatorics this is a difficult problem to solve so we do this by a brute force method looking at all the possibilities. (the downfall of this method is that a possibility can be overlooked)

There are 3 possible ways of distributing: 1. Balls are all in red boxes, 2. Balls are all in yellow boxes and 3. Balls are distributed between red and yellow boxes.

1. (3,0,0|0,0,0,0), (2,1,0|0,0,0,0), (1,1,1|0,0,0,0)
2. (0,0,0|3,0,0,0), (0,0,0|2,1,0,0), (0,0,0|1,1,1,0)
3. (2,0,0|1,0,0,0), (1,0,0|2,0,0,0), (1,1,0|1,0,0,0), (1,0,0|1,1,0,0)

There are a total of 10 ways to partition. Not only do we know how many partitions of N =32 into a partition of depth 7, but also it is easy to calculate the actual partitions since we add either 3+5*x or 2+5*y to the basis partition:

1. (18,3,3,2,2,2,2), (13,8,3,2,2,2,2), (8,8,8,2,2,2,2),
2. (3,3,3,17,2,2,2), (3,3,3,12,7,2,2), (3,3,3,7,7,7,2),
3. (13,3,3,7,2,2,2), (8,3,3,12,2,2,2), (8,8,3,7,2,2,2), (8,3,3,7,7,2,2)

I will call partitions of type 1 and 2 isolated or non-bonding partitions. Partitions of type 3 will be named mixed or bonding partitions. Several partitions can only be of the non-bonding type. Examples are partitions into depth k = 3 and k = 4. The basis representations for N = 3 mod 5 and k = 4 are (2,2,2,2) and for N = 2 mod 5 (3,3,3,3). The number of objects is (N-8)/5 and (N-12)/5 and they can only be isolated partitions.

This partition is found in sequence (OEIS A001400) [1,1,2,3,5,6,9,11,15,..]. It is the number of partitions of integers 5n + 8 or 5n + 12 into 4 parts (+/-) 3 mod 5. As given by example for N = 28, n = 4 and there are 5 partitions of 28: (7,7,7,7), (12,7,7,2), (12,12,2,2), (17,7,2,2) and (22,2,2,2}. This is equivalent to the number of partitions of n objects into at most 4 parts.

At depth k = 3 we can show that (OEIS A001399) [1,1,2,3,4,5,7,8,10,12…] is the number of partitions of integers 5n+6 and 5n+9 into 3 parts (+/-) 3 mod 5. As given by example for N = 46, n = 8 and there are 10 partitions (non-bonding) of basis (2,2,2) of 46 into 3 parts:

42	2	2
37	7	2
32	12	2
32	7	7
27	12	7
27	17	2
22	22	2
22	17	7
22	12	12
17	17	12

This is equivalent to the number of partitions of n objects into at most 3 parts.

Many other sequences can be generated using this method for partitioning integers into k parts. For example, in the distribution of n object into 3 red boxes and 4 yellow boxes the total number of bonding and non-bonding partitions increase as 2,5,10,19,32,54 and the number of bonding partitions is 0, 1, 4 10, 21, 38 (OEIS A009898). This later sequence has a topological application in chemistry. It is equivalent to the number of atoms in shell k bonded to shell (k-1) in certain forms of zeolites.

At depth 3 (OEIS A001399) and depth 4 (OEIS A001400) we can find sequence relations between the partitions. For example, for I = 0,1,2,3 ..

[1] p(6i+2,3)-p(6i+1,3) = p(6i+3,3)-p(6i+2,3) = p(6i+4,3)-p(6i+3,3) = p(6i+5,3)-p(6i+4,3) = i+1

[2] p(6i,4) - p(6i+1,4) - p(6i+4) + p(6i+5,4) = i+1.

Equating both sides shows a sequence relation between the depth 3 and depth 4 for partitions into parts (+/-) 3mod5:

[3] p(6i+2,3) - p(6i+1,3) = p(6i,4) - p(6i+1,4) - p(6i+4) + p(6i+5,4)

Ramanujan Type Congruences

Almost 100 years ago Ramanujan discovered several congruences for the unrestricted partition of integers. For example,

[4] p(5j+4) = 0 (mod5)

These congruences indicate that for some given (mod ℓ) the partition sequence is periodic. These congruences are typically of the form:

[5] p(Aj+B) = 0 (mod ℓ) $j \geq 0$

The congruences for the restricted partitions of Ramanujan's second identity can be found using the techniques mentioned above. These congruences can be written in a similar form to [5]:

[6] p(Aj+B,m) = 0 (mod ℓ)

or in tabular form with the parameters A,B,m, and ℓ.

Some simple parameters are derived from the non-bonding basis representations for m = 3, 4 and 5 (e.g.) (2,2,2) or (3,3,3); (2,2,2,2) or (3,3,3,3); and the sum (2,2,2,2,2) and (3,3,3,3,3) which are partitions of (+/-) 4mod 5, (+/-) 3mod 5, and (+/-) 0 mod 5 (two basis representations are required).

The following Table summarizes some known congruences:

A	B[1]	m	ℓ		A	B	m	ℓ
18	3	3	3		180	2	5	3
30	5	3	5		300	3	5	5
42	6	3	7		420	12	5	7
36	3	4	3		660	15	5	11
60	4	4	5		780	13	5	13
84	18	4	7		540	29	5	3^2
132	7	4	11		1500	48	5	5^2

[1] B is based on the first value for which p(B,m) = 0 (mod ℓ)

The congruences for m = 3 and m = 4 can be checked with the sequences OEIS A001399 and A001400. The congruences for m = 5 are obtained from A001401 where the number of partitions for integers 0 mod 5 is the sum of A001401(n) + A001401(n+1). [2,3,5,8,12,17,23,31,41..].

Kronholm (1) discusses a similar restriction Corollary to [6] which is modified for the basis representation for m =5:

Corollary: For ℓ and odd prime, j > 0, and $0 \le k \le (\ell-3)/2$ and $\alpha \ge 1$,

$P(\mathrm{lcm}(\ell) * \ell^{\alpha} * j - k\ell, \ell) = 0 \pmod{\ell^{\alpha}}$

Where lcm(ℓ) is the least common multiple of the numbers from 1 to ℓ.

This corollary works with the number of partitions for integers 0 mod 5 when ℓ = m = 5 and j >0.

Also, from [2] and the partition for m = 5, I obtain a modular equation in n,

[7] p(6n,4) - p(6n+1,4) + p(6n,5) + 2*p(6n+5,4) - p(6n+5,5) - 1 = 0 (mod n)

Example n = 8, p(48,4) - p(49,4) + p(48,5) + 2*p(53,4) – p(53,5) -1

= 1033 - 1089 + 6300 + 2*1350 - 8935-1

= 8 = 0 (mod 8)

For integers (+/-) 3 mod 5 and m = ℓ = 5 the lcm(ℓ)* ℓ^{α} does not exactly hold for the periodic sequence length. The basis representation for these integers is {3,3,2,2,2) and (3,3,3,2,2) which results in the sequence given by A097701. {1,2,5,9,16,25,39,56,80…}. Equation [6] becomes:

[8] p(30j +2) = 0 (mod 5)

However, given integers (+/-) 1 mod 5 and m = ℓ = 5 the period lcm(ℓ)* $\ell^{\alpha-1}$ agrees with the original corollary and integer sequence A002621 applies: {1,2,4,7,12,18,27,38,53,71….}:

[9] p(60j +12) = 0 (mod 5)

The Basis Representation and Generating Functions

An interesting relationship is found between the basis representations for each depth k and the generating function of the series. The generating function is a polynomial in the variable x which results in the integer sequence. I demonstrated in Chalkboard 9 how the Perrin sequence is a polynomial in x,

$F(x) = 3 + 0*x + 2*x^2 + 3*x^3 + 2*x^4 + 5*x^5 + 5*x^6 \ldots.$

This series is generated from the generating function

$A(x) = (3-x^2)/(1-x^2-x^3)$

Partitions are generated from similar reducible polynomials in $1/((1-x)^a*(1-x^2)^b*(1-x^3)^c\ldots.)$ where a, b and c are integers $\ge$ 0.

As an example I show above that the number of partitions of an integer 5n+8 into 4 parts (+/-) 3 mod 5 is given by the number of partitions of a number into at most 4 parts (OEIS A001400). The generating function and series is given by:

$F(x) = 1+x+2x^2+3x^3+5x^4+6x^5+9x^6+11x^7\ldots\ldots = A(x) = 1/((1-x)*(1-x^2)*(1-x^3)*(1-x^4))$

The original basis representation for this series is (2,2,2,2). Notice that the coefficients a,b,c.. in the denominator of A(x) are (1,1,1,1).

As another example the number of partitions of an integer 5n+12 into 5 parts (+/-) 3 mod 5 is given by (OEIS A097701). The generating function and series is given by:

$$F(x) = 1+2x+5x^2+9x^3+16x^4+25x^5+39x^6+56x^7\ldots\ldots = A(x) = 1/((1-x)^2*(1-x^2)^2*(1-x^3))$$

The original basis representation for this series is (33222). Notice that the coefficients a,b,c.. in the denominator of A(x) are now (2,2,1). In this case the basis can also be written as (3,2,3,2,2)

I can make the following rule for writing the generating function A(x) for any basis representation:

Given a basis representation of depth k, (3,3,3,….2,2,2….) composed of i '3s' and j '2s' where j+i = k the basis is rearranged first in alternating pairs of '3,2' and with the remaining '3s' or '2s' placed at the end, (3,2,3,2,3,2…..2,2,2) or (3,2,3,2,3,2…..3,3,3).

For example, for k = 9 one basis for an integer 4 mod 5 is (3,3,3,3,3,3,2,2,2) and can be written as (3,2,3,2,3,2,3,3,3).

For each pair (3,2) the corresponding coefficient a,b,c, is given the value of 2 and the remaining coefficients are a series of 1's. The sum of the coefficients a,b,c,.. is k. The resulting inverse polynomial is a generating function for the given basis representation.

In the example

$$A2(x) = 1/((1-x)^2*(1-x^2)^2*(1-x^3)^2*(1-x^4)*(1-x^5)*(1-x^6))$$

If the number of basis representations for a given k is greater than one as given by the Table above for the Number of Basis Representations then all basis representations and generating functions must be added to calculate the corresponding integer series for the partition k. The addition requires a shift by one for each successive representation. A shift is made by multiplying A(x) by x.

In the example k = 9 = 4 mod 5 (N =1) and all integers N' mod 5 require 2 basis representations. The first representation for an integer 4 mod 5 is (3,2,2,2,2,2,2,2,2). Since this is the first representation the successive generating function A2(x) is multiplied by x;

$$A1(x) = 1/((1-x)^2 *(1-x^2) *(1-x^3) *(1-x^4) *(1-x^5) *(1-x^6) *(1-x^7) *(1-x^8))$$

The resulting generating function is then

$$A1(x) + x*A2(x) = \text{polynomial in powers of } x$$

For r basis representations:

$$A1(x) + x^1 *A2(x)\ldots\ldots\ldots\ldots\ x^{r-1}*Ar(x) = \text{polynomial in powers of } x$$

In this example r = 2, *Mathematica(2)* gives the following result:

```
CoefficientList[
 Series[1 / ((1 - x) ^ 2 * (1 - x ^ 2) * (1 - x ^ 3) * (1 - x ^ 4) * (1 - x ^ 5) * (1 - x ^ 6) * (1 - x ^ 7) * (1 - x ^ 8)) +
   x / ((1 - x) ^ 2 * (1 - x ^ 2) ^ 2 * (1 - x ^ 3) ^ 2 * (1 - x ^ 4) * (1 - x ^ 5) * (1 - x ^ 6)), {x, 0, 30}], x]
```

Integer sequence:
{1,3,6,12,22,38,63,102,158,240,355,515,733,1029,1419,1935,2604,3466,4563,5955,

7695,9868,12551,15851,19877,24774,30678,37783,46273,56386,68365}

The number of partitions of an integer 5n + 19 into 9 parts (+/-) 3 mod 5 is given by the above series.

For n = 2 the integer 5n +19 =29 the A(2) = 6 partitions of 29 are [13,2,2,2,2,2,2,2,2], [12,3,2,2,2,2,2,2,2],[8,7,2,2,2,2,2,2,2],[7,7,3,2,2,2,2,2,2], [8,3,3,3,3,3,3,3,3], and [7,3,3,3,3,3,3,2,2].

This method applies to the partitioning of A003106(N) = p(N,1) + p(N,2) + p(N,3) + ……. + p(N,k). An analogous partitioning of A003114(N) follows the general rules and algorithm applied here. It can be shown that for the first Rogers-Ramanujan identity that for r basis representations:

$A11(x) + x^3 * A21(x) \ldots\ldots\ldots\ldots X^{3(r-1)} * Ar1(x)$ = polynomial in powers of x

Where A11(x),A21(x)…Ar1(x) are the corresponding generating functions.

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1. B. Kronholm, *On Congruence Properties of Consecutive Values of P(N,M)*, Electronic Journal of Combinatorial Number Theory, 7, (2007) #A16
2. Mathematica, Wolfram Research, Version 10.4