Chapter 18

A General Algorithm for Restricted Partitions, Perrin and Padovan Sequences -

In the previous Chapter I discussed partitions of the Rogers Ramanujan identities which had a special property that they could be obtained from generating functions of special basis representations. The number of bonding pairs of two numbers, {3-2} or {4-1} of the partitions (+/-) 3 mod 5 or (+/-) 1 mod 5 in terms of reducible polynomials is $1/((1-x)^a*(1-x^2)^b*(1-x^3)^c....)$ where a, b and c are integers ≥ 0 that are ***expanded by assigning the corresponding coefficient a,b,c, the value of 2 and the remaining coefficients a series of 1's.***

In this Chapter this definition is generalized to any partition of two numbers mod(ℓ) and further rules are shown for **any number** of bonding pairs mod(ℓ). The following serve as examples:

a. Find the number of partitions into 4 parts of (+/-)2 mod 7 for an integer 4 mod 7. The prime basis is (5,2,2,2) of weight 11. Let (5-2) be the bonding pair, then the generating function is $F(x) = 1+2x+4x^2+7x^3+11x^4+16x^5+...... = A(x) = 1/((1-x)^2*(1-x^2)*(1-x^3))$. For an integer 7n + 11 = 32 = 4 mod 7; a(3) = 7 partitions: [26,2,2,2],[23,5,2,2],[16,9,5,2],[9,9,9,5],[19,9,2,2],[16,12,2,2] and [12,9,9,2]. [OEIS A000601]
b. Find bonding partitions for 3 parts (I,j,k) mod 5. Let i = 4, j = 2, k = 1. The weight of the basis {4,2,1} is 7. The coefficient **a** for a 3 bonded partition is **a** = 3;
$F(x) = 1+3x+6x^2+10x^3+15x^4+21x^5+...... = A(x) = 1/((1-x)^3$
For the integer 5n +7 = 17; a(2) = 6 partitions: [14,2,1],[12,4,1],[11,4,2],[9,7,1],[7,6,4],[9,6,2] [OEIS A00217]
c. Find the number of partitions into 6 parts mod 5 as basis [4,4,3,3,1,1]. This prime basis has weight 16. There are two bonding partitions [4-3-1] with the generating function;
$F(x) = 1+3x+9x^2+19x^3+39x^4+69x^5+...... = A(x) = 1/((1-x)^3*(1-x^2)^3))$. In this and the previous case all partitions were bonding. [OEIS A038163]

The above examples illustrate a general algorithm for calculating partitions of any combination of integers of a given modulus, weight and length.

Let $(a_1,a_2,a_3,....a_k)$ be a prime basis of k terms with weight $N_o= a_1+a_2+a_3+...a_k$. Let m_i be the number of bonding terms of various lengths $(a_1,a_2,..a_j)> (a_1,a_2,..a_{j-1})> (a_1,a_2,..a_{j-2})$ and n_i the number of remaining non-bonding terms $(a_{n1},a_{n2},.....a_{ni})$ where all integers are integers mod(ℓ).

Then the generating function is F(x):

$$F(x) = \sum_{(i=0)} P_k(n)\, x^{i} = A(x) = 1/((1-x)^{m1}*(1-x^2)^{m2..}....(1-x^i)^{mi}*(1-x^{j+n1}).......(1-x^{j+ni}))$$

With $m_1+m_2+...m_i+n_1+....n_i = k$ and $P_k(n)$ is the number of partitions of length k for a series of integers $\ell*n + N_o$ with $n \geq 0$.

d. Find the number of partitions of length 10 in parts 4 mod 5, 3 mod 5, 2 mod 5 and 1 mod 5 where the weight of the prime basis is 21.
A basis can be written as {4,3,3,2,2,2,2,1,1,1} and arranged into bonding and non-bonding pairs as {(4,3,2,1},(3,2,1),(2,1),2}. From the algorithm the generating functions is

$$A(x) = 1/((1-x)^4*(1-x^2)^3*(1-x^3)^2*(1-x^4)^1)$$

With 4+3+2+1 = k = 10.

Mathematica gives the following series:

{1,4,13,34,80,170,339,636,1141,1964,3270,5280,8309,12768,19219,28382, 41209,58898,82994,115410,158561,215398,289579,385520,508614,665300,863339,1111910,1421949, 1806278,2280022}

The number of partitions for n = 3, integer 5*3+21 = 36 is $P_{10}(3)$ =34

The partitions are shown in the following Tables:

Partitioning of 36 into 10 parts with the basis representation {4,3,3,2,2,2,2,1,1,1}

Number\Basis	4	3	3	2	2	2	2	1	1	1
1	19	3	3	2	2	2	2	1	1	1
2	4	18	3	2	2	2	2	1	1	1
3	4	3	3	17	2	2	2	1	1	1
4	4	3	3	2	2	2	2	16	1	1
5	4	13	8	2	2	2	2	1	1	1
6	4	3	3	12	7	2	2	1	1	1
7	4	3	3	2	2	2	2	11	6	1
8	4	13	3	7	2	2	2	1	1	1
9	4	13	3	2	2	2	2	6	1	1
10	4	8	3	12	2	2	2	1	1	1
11	4	8	3	2	2	2	2	11	1	1
12	4	8	8	7	2	2	2	1	1	1
13	4	8	8	2	2	2	2	6	1	1
14	4	8	3	7	7	2	2	1	1	1
15	4	8	3	2	2	2	2	6	6	1
16	4	3	3	7	2	2	2	6	6	1
17	4	3	3	12	2	2	2	6	1	1
18	4	3	3	7	2	2	2	11	1	1
19	9	8	3	2	2	2	2	6	1	1
20	4	3	3	7	7	2	2	6	1	1
21	4	3	3	2	2	2	2	6	6	6
22	14	3	3	2	2	2	2	6	1	1
23	14	8	3	2	2	2	2	1	1	1
24	14	3	3	7	2	2	2	1	1	1
25	4	3	3	7	2	2	2	6	6	1
26	9	8	8	2	2	2	2	1	1	1
27	9	3	3	7	7	2	2	1	1	1
28	9	3	3	2	2	2	2	6	6	1
29	9	13	3	2	2	2	2	1	1	1
30	9	3	3	12	2	2	2	1	1	1
31	9	3	3	2	2	2	2	11	1	1
32	4	3	3	7	7	7	2	1	1	1
33	9	8	3	7	2	2	2	1	1	1
34	9	3	3	7	2	2	2	6	1	1

The Perrin Sequence, Padovan Sequence and Partitions

Let a(0) = 3, a(1)=0, a(2)=2 and expand the Perrin sequence as a composition of 3s and 2s;

(3),(0),(2),(3),(2),(3,2),(3,2),(3,2,2),(3,2,3,2),(3,2,3,2,2),(3,2,3,2,3,2,2),(3,2,3,2,3,2,3,2,2)…..

If each term is expressed as the number of bonding terms (3,2) plus a constant the series can be written as:

Perrin Sequence as Multiples of Number of Bonding Pairs and additive constant			
5*0+3	5***1**+0	5***5**+4	5***28**+18
5*0+0	5***1**+2	5***7**+4	5***37**+24
5*0+2	5***2**+0	5***9**+6	5***49**+32
5*0+3	5***2**+2	5***12**+8	5***65**+42
5*0+2	5***3**+2	5***16**+10	5***86**+56
5***1**+0	5***4**+2	5***21**+14	5***114**+74

If we start with the first bonding pair 5 = (3,2) the 5th number of the sequence, the number of bonding pairs increases as:

[1] 1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49….

This is a sequence with p(0) = 1, p(1)=1, p(2)=1 and p(n) = p(n-2)+p(n-3) known as the Padovan sequence.

The actual sequence is defined from the initial conditions p(0) = 1, p(1)=0, p(2)=0.

[2] 1,0,0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49….

The composition of 3's and 2's in the original Perrin sequence is then related to the Padovan sequence. For example a(13) = 39 = (3,2,3,2,3,2,3,2,3,2,3,2,3,2,2,2) is composed of 16 numbers which is the 16th term of the Padovan sequence [2] or the 11th term in [1].

The Padovan sequence p(N) also is the number of distinct ways (order dependent) of composing the (N-3)rd term. For example p(12) = 5 with the number (12-3) = 9 composed 5 distinct ways: (3,2,2,2), (2,3,2,2),(2,2,3,2), (2,2,2,3), (3,3,3).

Many of these partitions are described in OEIS A000931.

In Chapter 17 I mentioned isomorphic recurrence sequences of the Perrin sequence. The difference of the first isomorphic sequence from the Perrin sequence results in the series {5,0,5,5,5,10,10,15,20,25..}. If we divide these numbers by 5 we get the Padovan series 1,0,1,1,1,2,2,3,4,…[OEIS A182097]. I showed that:

[3] P(n) =[A(n+1) – A(n-23)]/5

with A(n) = A(A099559) from OEIS generated from the binomial coefficients

[4] $$A(n) = \sum_{k=0}^{i} \binom{n-4k}{k+1}$$

with $0 \leq i \leq (n-1)/5$.

An interesting transform, INVERT and INVERTi relate the Fibonacci series of polynomials with the Padovan polynomial series. Let the Fibonacci generating function be shifted multiplying by x^2 and adding these leading terms to give:

[5] $$F(x) = 1x + 1x^2 + 1x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} \ldots\ldots$$

then the inverse transform INVERTi F(x)) = -1/(1+F(x)) + 1 = P(x)

[6] $$P(x) = 0 + 1x + 0x^2 + 0x^3 + 1x^4 + 0x^5 + 1x^6 + 1x^7 + 1x^8 + 2x^9 + 2x^{10} + 3x^{11} \ldots$$

results in the generating function of the Padovan numbers.

The algebraic transform from Padovan to Fibonacci is INVERT: P(x)) = 1/(1-P(x)) -1 = F(x)

The Padovan series can also be extended to negative numbers with:

[7] Pi(n) = Pi(n+3) – Pi(n+1)

Initiating with Pi(0) =1, Pi(-1)=0 and Pi(-2) = 1 the negative series is:

[8] 1,0,1,0,0,1,-1,1,0,-1,2,-2,1,1,-3,...

It can be shown that the algebraic transforms between the Padovan polynomial and the negative Padovan polynomial are:

[9] P'(x) = 1/(1-Pi(x)) -1

[10] Pi(x) = -1/(1+P'(x)) + 1

With $Pi(x) = x + 0*x^2 + 0*x^3 + x^4 - x^5 + x^6 + 0*x^7 - x^8 \ldots$

And $P'(x) = x + x^2 + x^3 + 2x^4 + 2*x^5 + 3*x^6 + 4*x^7 + 5*x^8 \ldots$

The algebraic equation between these polynomials is

[11] P'(x)*Pi(x) - (P'(x) - Pi(x)) = 0

[12] F(x)*P(x) - (F(x) - P(x)) = 0

with the polynomials P(x), F(x) and P'(x) defined above.

In a recent paper C. Ballantine and M. Merca [***1***] show that the Fibonacci and the Padovan sequences can be expanded as sums of Multinomial coefficients. These coefficients are integer partitions of n into odd parts where n is the n^{th} index of the Fibonacci series F_n or of the Padovan sequence P_n. The difference is the partitioning of the Padovan sequence into odd parts having no part equal to one.

[13] $$P_n = \sum_{a2t2+a3t3+\cdots+a\left[\frac{n}{2}\right]t\left[\frac{n}{2}\right]=n}^{k} \binom{t2+t3+\cdots\ldots\ldots+t[\frac{n}{2}]}{t2,t3,\ldots\ldots t[\frac{n}{2}]}$$

with $a_k = 2k-1$, ( ) is the multinomial coefficient and [x] represents the smallest integer not less than x.

As an application calculate the n = 19th term in sequence [2] above using [13]. In this case k = 2 ….. 10 and 19 is partitioned into odd parts as follows:

t_k \ a_k	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**Multinomial**
	t2	**t3**	**t4**	**t5**	**t6**	**t7**	**t8**	**t9**	**t10**	**Coefficient [a]**
1	0	0	0	0	0	0	0	0	1	1
2	2	0	0	0	0	1	0	0	0	3
3	1	1	0	0	1	0	0	0	0	6
4	1	0	1	1	0	0	0	0	0	6
5	0	1	2	0	0	0	0	0	0	3
6	0	2	0	1	0	0	0	0	0	3
7	3	2	0	0	0	0	0	0	0	10
8	4	0	1	0	0	0	0	0	0	5
									SUM P_{19}	**37**

a. $(\sum tk)!/(t1! * t2! * \ldots\ldots tk!))$

Each Padovan number is then a sum of the number of ways of partitioning n into odd parts not including the number 1. In this example there are 8 ways (19), (13,3,3), (11,5,3), (9,7,3), (7,7,5), (9,5,5), (5,5,3,3,3), and (7,3,3,3,3). The values of these multinomial coefficients add to the corresponding nth Padovan number.

Proposition 1: The nth term of the Padovan sequence derived from the number of bonding pairs of the Perrin sequence is related to the number of ways of partitioning an integer into odd parts not including the number 1(OEIS A087897). ***That is, the nth term of OEIS A087897 is the number of multinomial coefficients and the sum of the values of the multinomial coefficients is the Padovan number!***

In their paper Ballantine and Merca show an identity which relates the Padovan numbers [2] and the inverse Padovan numbers [8]:

[14] $$P_n = \sum_{j=0}^{k} \binom{k}{j} P_{n-2k-j}$$

where k is the number of multinomial coefficients. As example let n = 20 then k= A087897(20) = 10. The RHS is then the product of the binomial coefficients $\binom{10}{j}$ as j increases from 0 to 10 and the Padovan numbers P_0, P_{-1},P_{-2} ……P_{-10} shifted sequence [8] with P(0) = 1, P(-1) = -1 …..P(-10) = 4. The result can be shown as the sum:

$$1*1 – 1*10 +1*45 + 0*120 -1*210 + 2*252 -2*210 +1*120+1*45 – 3*10 +4*1 = 49 = P_{20}$$

Using Basis Representations for the Calculation of Padovan Numbers from Multinomial Coefficients

Examination of the table above for values of the multinomial coefficient suggests that the value depends on the particular partition length and weight. If we look at the partitions that produced a coefficient value of 3 (t_2, t_5, and t_6) they all have the form (2,1) so partition into 3 parts: (3,3,13), (7,7,5) and (5,5,9) maintaining all parts are odd.

Consider a basis representation for these partitions of (3,3,3), (2,2,0) and (0,0,4). Note that the weights of these basis are all equal to 4 mod 5 also the modulus of N. If we calculate the number of objects j = (N- weight)/5 then for N = 19, j = 2,3,3, respectively. As in our discussion above of applying these object to the basis representation we get (3,3,5***2**+3), (5***1**+2, 5***1**+2, 5***1**+0), and (5***1**+0, 5***1**+0, 5***1**+4) = (3,3,13), (7,7,5) and (5,5,9) as required. Note that if the objects were distributed differently the partition would not be allowed since an even number such as 6, 10 or 12 would be calculated for the partition.

In principle all partitions can be formed from a basis mod 5. The remaining basis in the above table for (t_1, t_3, t_4, t_7, t_8, and t_8) are (4), (3,1,0), (3,2,4), (3,3,3,0,0), and (3,3,3,3,2) with j = 3, 3, 2 and 1. The first basis then represents (5***3**+(4)).

Note that there are only odd lengths which are possible for each partition since for odd numbers $\boldsymbol{a}$ and even numbers $\boldsymbol{e}$; $\boldsymbol{a}+\boldsymbol{a}+\boldsymbol{a} = \boldsymbol{a}$ and $\boldsymbol{a} + \boldsymbol{a} = \boldsymbol{e}$. If N is odd then all partitions have an odd length 1,3,5,7,… and if N is even the partitions are of length 0,2,4,6,8,….

Once all the possible basis representations are written down for N then the multinomial coefficient can be interpreted from the basis since each part of the basis represents a number. For example (3,3,3,3,2) = { 4,1} $\underline{\underline{M}}$ 5!/(4!*1!) = 5. I will use the symbol $\underline{\underline{M}}$ to represent transformation of the representation to a multinomial coefficient. The above example for N = 19 gives:

(4) $\underline{\underline{M}}$ 1

(3,3,$\underline{3}$) $\underline{\underline{M}}$ 3

(3,1,0) $\underline{\underline{M}}$ 6

(3,2,4) $\underline{\underline{M}}$ 6

(2,2,0) $\underline{\underline{M}}$ 3

(0,0,4) $\underline{\underline{M}}$ 3

(3,3,3,0,0) $\underline{\underline{M}}$ 10

(3,3,3,3,2) $\underline{\underline{M}}$ 5

SUM = P_{19} = 37

[Note that (3,3,$\underline{3}$) and (3,3,3) have a separate designation and transform as (3,3,$\underline{3}$) $\underline{\underline{M}}$ 3 and (3,3,3) $\underline{\underline{M}}$ 6]

The above basis can be expressed as (A), (AAB), (ABC), (AAABB), and (AAAAB) with each appearing 1,3,2,1, and 1 times. The total number of the appearances is 8. The sum of the transformed basis representations to multinomial coefficients is the nth Padovan number.

The sum of the different representations are the number of ways of partitioning a number into odd parts not including the number 1. Transform these basis representations to multinomial coefficients and sum to get the Padovan number.

Other rules can be derived based on the relationship of the basis partitions to generating functions.

For example, given an even N = 1 mod 5 (e.g. 6,16,26,..) the weight of a two part partition is 6 and these partitions are (3,3), (4,2) and (5,1). Based on the representative generating function non-bonding (3,3) and bonding (4,2) and (5,1):

$$A((3,3)) = 1/((1-x)*(1-x^2)) = 1,1,2,2,3,3,4,4,5,5..$$
$$A((4,2)) = 1/((1-x)^2) = 1,2,3,4,5,6,7,8,9....$$
$$A((5,1)) = 1/((1-x)^2) = 1,2,3,4,5,6,7,8,9....$$

These 3 terms then sum as 3,6,8,11,13,16,18,23.. and it can be shown that for N = 26 there are 6 basis representations of 2 parts (11,15), (9,17), (19,7), (21,5), (23,3) and (13,13). Note that the first 5 transform to a multinomial coefficient of 2 and the last to 1.

In a future Appendix I will show the relations between equations [3] and [4] derived from the isomorphic Perrin equations and the multinomial expansion of the Padovan equation [13].

1. Ballantine, C., and Merca, M., *Padovan Numbers as sums over Partitions into odd parts*, Journal of Inequalities and Applications, 2016.

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