Geometry of the Perrin and Padovan Sequences

This chalkboard discusses some interesting geometric properties of the “plastic number” ψ = 1.324717..

This number, as discussed previously, is the real, irrational number which is the solution to the cubic equation $x^3 = x + 1$. In 1960 the Benedictine order monk and architect Dom Van der Laan considered this number to be an ideal geometric proportion similar to the ancient architecture proportion of the Golden Mean or Divine number ϕ = 1.618033.. = $(1+\sqrt{5})/2$. Examples of the Golden mean are found in the Greek Acropolis and Parthenon, and Romanesque and Gothic architecture. In art, *De divina proprtione* published in 1509 illustrated drawings by Leonardo da Vinci in which the golden ratio appears. Images of Dom Van der Laan’s architecture can be found on the web.

I have taken some information from the following references:

1. J. Aarts, R. Fokkink, and G. Kruijtzer. “Morphic numbers” NAW 5/2 (2001)
2. L. Marohnic, and T. Strmecki. “Plastic Number: Constructiion and Applications”, International Virtual Conference, http://www.arsa-conf.com (2012).
3. V.W. de Spinadel and A.R. Buitrago. “Towards van der Laan’s Plastic Number in a Plane”, Journal of Geometry and Graphics 13(2), (2009).

The numbers ψ and ϕ are the only two numbers which satisfy the definition of a *morphic number*: A real number p>1 is morphic if it satisfies the two relations,

[1] a.) $p + 1 = p^k$ b.) $p - 1 = p^{-j}$

where k and j are integers.

Van der Laan considered ψ to be a more natural scale for comparing objects. The Laurent type geometric sequence of increasing scale is;

[2] $,\psi^{-2}, \psi^{-1}, 1, \psi, \psi^2, \psi^3, \ldots\ldots..$

This sequence has an interesting property similar to the Perrin and Padovan sequences;

[3] $\psi^{\ell} = \psi^{\ell-2} + \psi^{\ell-3}$ or $\psi^{\ell} = \psi^{\ell-1} + \psi^{\ell-3}$

(ℓ a positive or negative integer).

It can be shown that ψ is a morphic number satisfying [1a] and [1b] with k = 3 and j = 2.

Based on the theorems of Euclid and Pythagoras the line segment of length ϕ can be constructed with a straight edge and compass. Given a unit length of 1 construct AB = 2 and a perpendicular to point B of length BC. Then a hypotenuse connecting AC has length $\sqrt{(2^2+1)} = \sqrt{5}$. Add the unit length and bisect the line to get ϕ.

In general, *if a real number α satisfies an irreducible polynomial over the field of rational numbers of degree k, and if k is not a power of 2 then α is not constructible.* (I.N. Herstein, **Topics in Algebra**, Blaisdell Publishing Co. 1964. pp 189).

As a corollary it is also impossible by straight edge and compass alone to trisect a 60° angle. Herstein shows that this would be equivalent to finding the root α of a degree 3 polynomial equation $4\alpha^3 - 3\alpha = \frac{1}{2}$. By the previous paragraph, α is not constructible.

So given that ψ is the real root of a cubic equation, we can construct this root by using methods from paper folding or origami. Several methods for doing this are available on the web. The construction of ψ by paper folding is associated with the trisection of an angle and the geometric series [2] for ψ^k.

Construction of ψ.

1. Given a cubic equation $x^3 - x - 1 = 0$ solve for x.
   It is convenient to use a new coordinate system on the paper that puts the line OT containing the beginning point O and the terminal point T at the left edge of the paper and perpendicular to the bottom edge of the paper. If the length OT = $\sqrt{5}$ then this edge can be constructed using a ruler and compass by making a right triangle with sides of length OV = 2 and TV = 1.

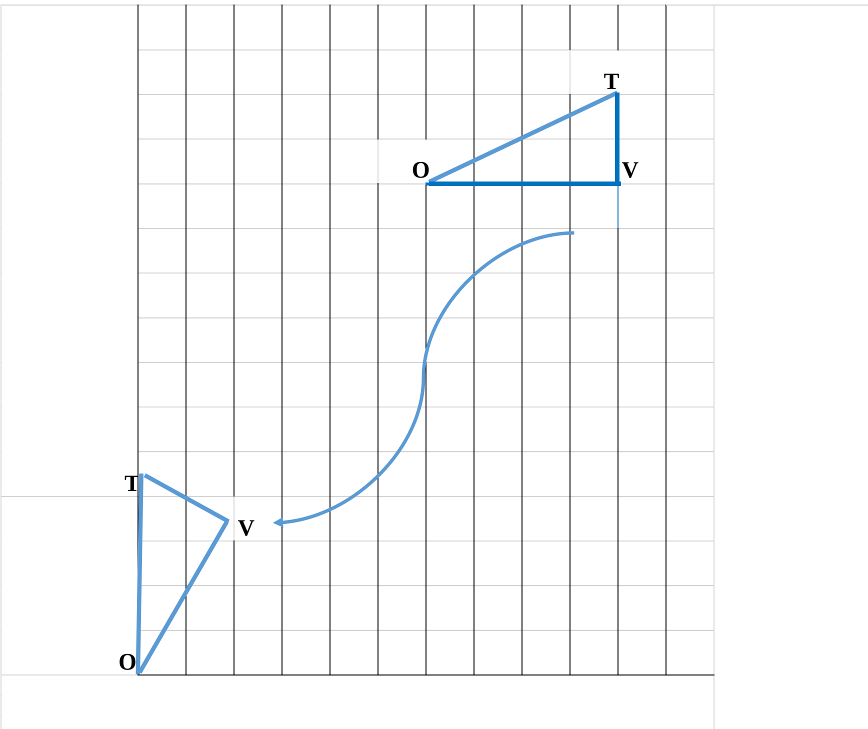


FIGURE 1

2. Draw a line from T along the edge of the right triangle and extend to the bottom of the paper. Mark point H one unit from the right angle on this line such that TH = 2. Mark a point A one unit (bisector) from the bottom edge of the triangle such that OA = 1.

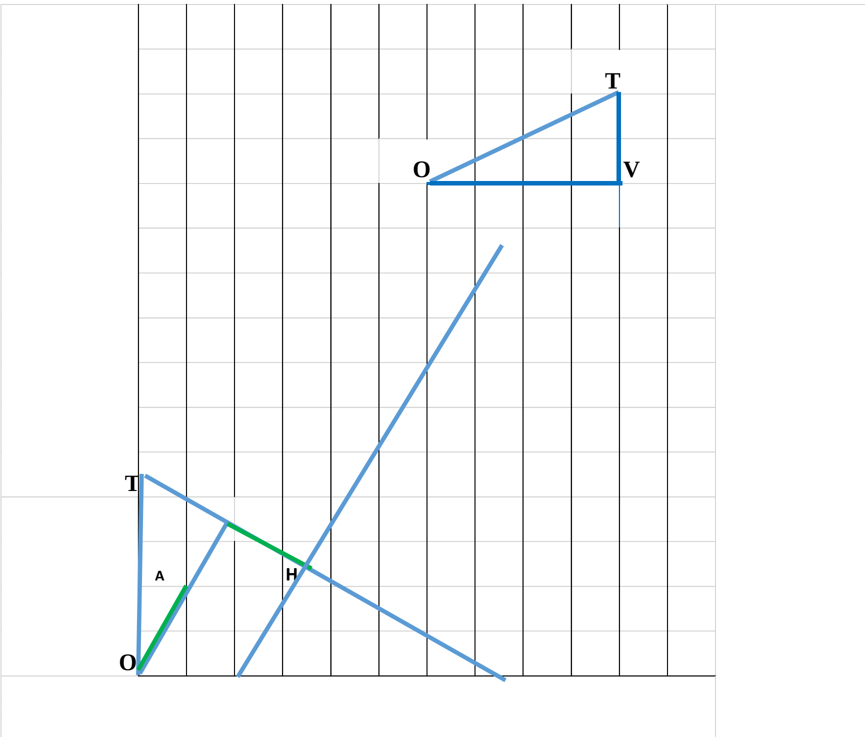


FIGURE 2

3. A key axiom to finding the solution to a cubic equation or trisecting an angle is that '*given two points such as O and T and two lines (lines perpendicular to point H) there is a fold that places O onto the line formed by the segment TH and places T onto a line perpendicular to the line containing the seqment TH*'.
   Label these points P1 and P2 respectively. Draw the fold line (red line in figure 3).

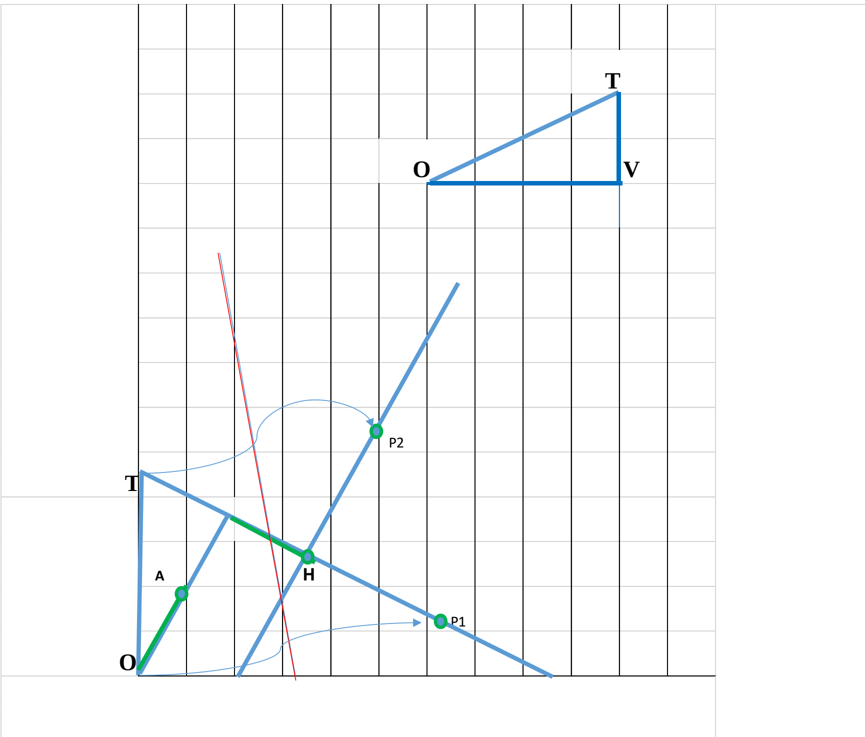


FIGURE 3

4. Since the fold line is a bisector of line segments TP2 and 0P1 it is also perpendicular to lines TP2 and OP1. These intersections are labeled B2 and B1. Connect the points A and B1.

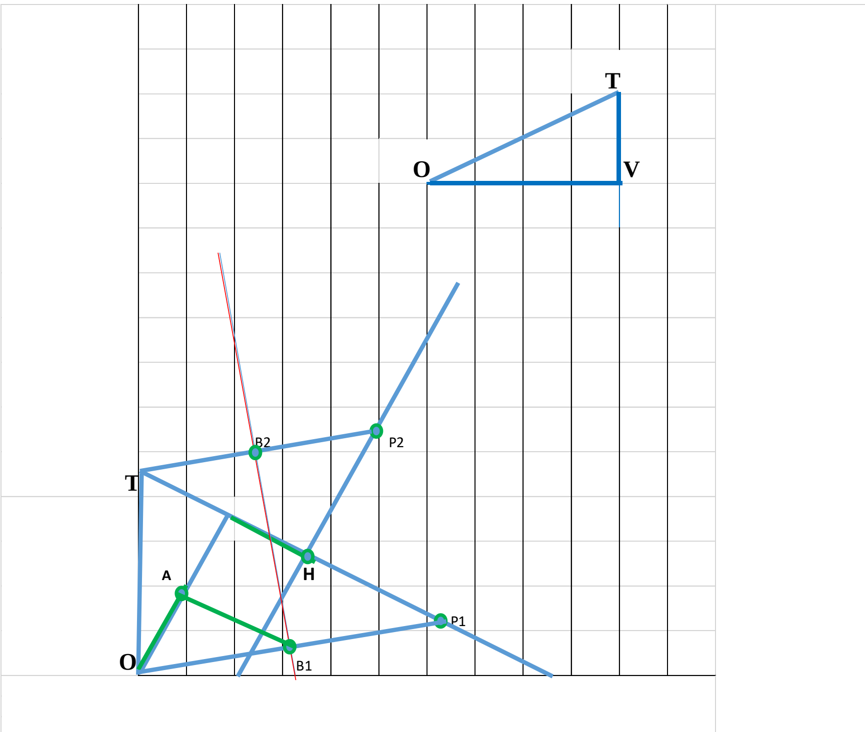


FIGURE 4

5. We have our first solution! The length of line segment AB1 = ψ. Some other geometric sequence lengths can also be found. Note the length TV = 1. If we draw a line from B1 and intersect segment HP1 at a right angle at H2 then the line segment TH2 has a length equal to 1 + ψ. But this according to the solution of the cubic, is exactly the length ψ^3. Since ψ is a morphic number the distance $\psi - 1 = \psi^{-4}$. The segment A2B1 = ψ^{-4}. (see figure 5)

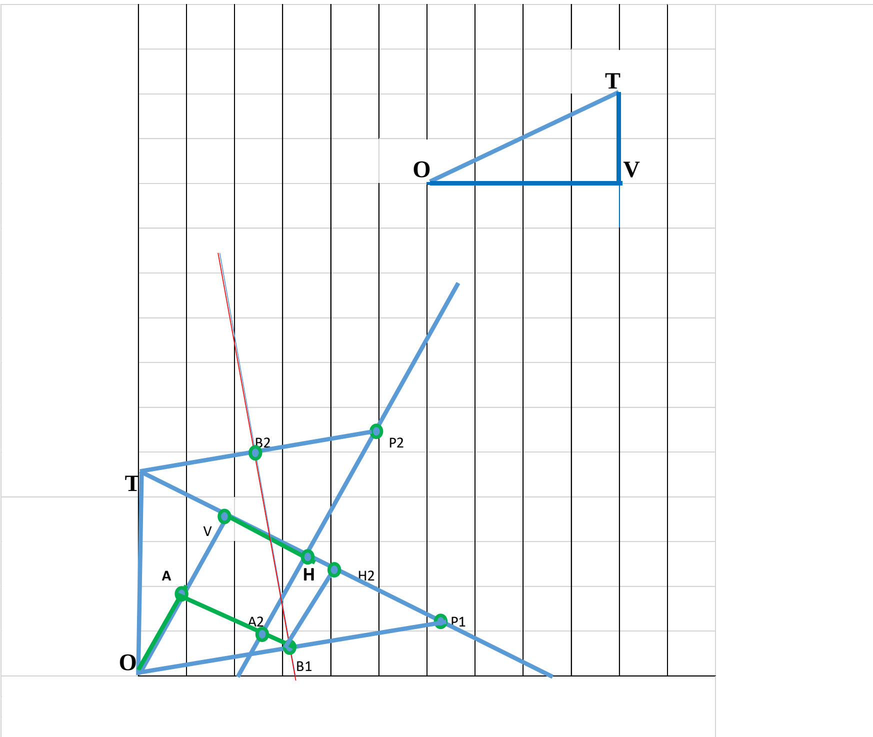


FIGURE 5

6. I have used a coordinate system for obtaining the trisection of an angle. I will show that the trisected angle is a power of ψ.
   Draw a line perpendicular to the x axis to point P1 to construct the segment CP1. Connect points P1 and P2 and use a compass to mark the length CP1 on line P1P2. Call these points P3 and P4. Draw lines OP3 and OP4. (see Figure 6).

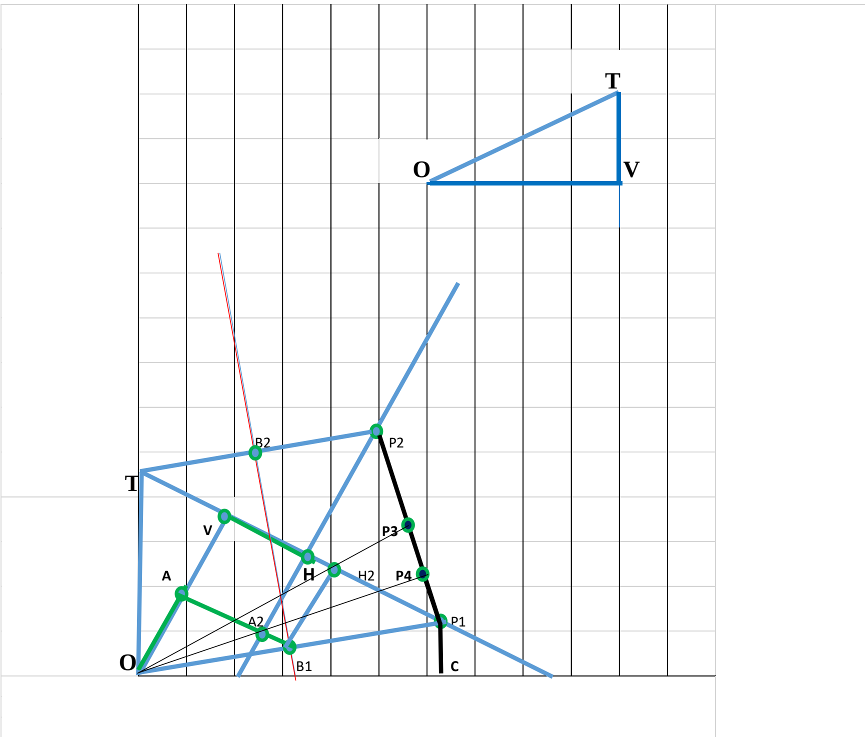


FIGURE 6

7. Since line P1P2 is constructed from the fold line (red) and the length P3P4 = P4P1, OP4 is a perpendicular bisector of the segment P1P2. Call ∠P1OC = γ and ∠P4OP1 = β. Then

$$\sin\gamma = \frac{CP1}{OP1} = \frac{P4P1}{OP1} = \sin\beta$$

and since triangle P3OP1 is isosceles ∠P3OP4 = α = ∠P4OP1 = β. Therefore $\beta = \alpha = \gamma$ and the angle ∠P3OC = $\alpha + \beta + \gamma$ is trisected! The following identities can be shown:

∠TOA = $\tan^{-1}(1/2)$

∠AOC = 90° - $\tan^{-1}(1/2)$

∠AOB1 = $\tan^{-1}(\psi/1)$

γ = ∠AOC - ∠AOB1

$\gamma = 63.43495^{o} - 52.95166^{o} = 10.483289^{o}$

Construct a unit length along the x axis from the origin. Then the perpendicular length PQ is

$\tan\gamma * 1 = 0.185037 = \Psi^{-6}$ and so if OS = Ψ then RS = Ψ^{-5} and if OS' = 1 + Ψ then RS' = Ψ^{-3}

(see figure 7).

Construct a diagonal from B1 thru point V. Extend a perpendicular of segment TH from point T. Then according to reference (3) the segment TW = 1/ Ψ.

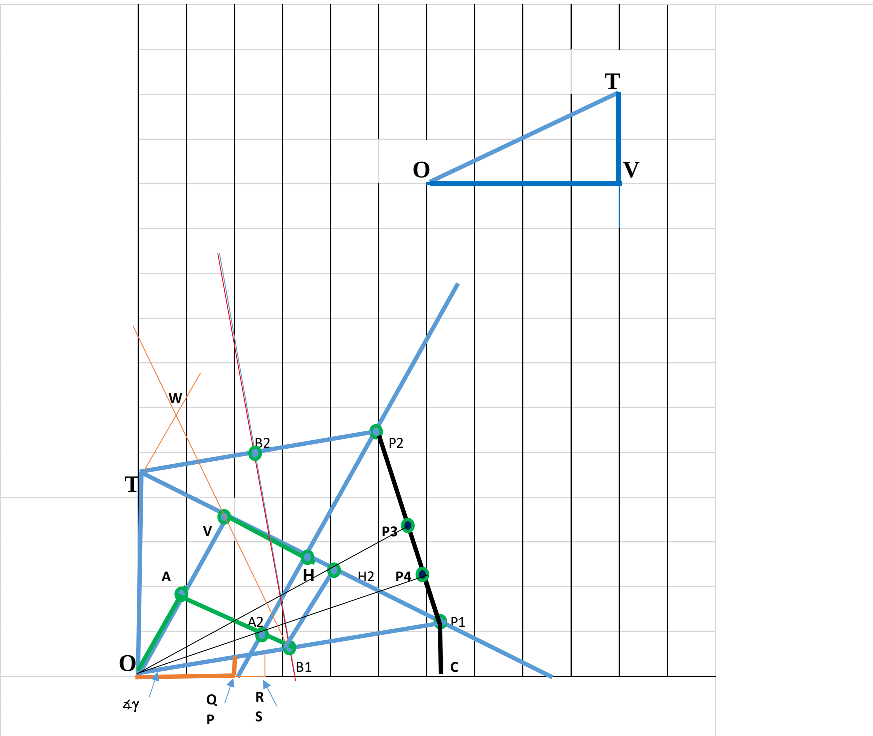


FIGURE 7

From the above figures the paper folding method has constructed segments of proportional length

ψ, 1/ ψ, ψ^{3}, 1/ ψ^{6}, 1/ ψ^{5} and 1/ ψ^{4}. From the Laurent geometric sequence all other lengths are constructible.

$\psi^{2} = \psi + 1/\psi$

$\psi^{-2} = \psi^{-4} + \psi^{-5}$

$\psi^{4} = \psi + \psi^{2}$

$\psi^{-3} = \psi^{-5} + \psi^{-6}$

$\psi^{-7} + \psi^{-6} = \psi^{-4}$ implies $\psi^{-7} = \psi^{-4} - \psi^{-6}$

$\psi^{5} = \psi^{2} + \psi^{3} = \psi^{2}(1 + \psi) = \psi^{2} * \psi^{3}$

Building a Perrin Sequence

The construction of the plastic number using origami leads to the following theorem:

19.1 Theorem: ***The Perrin sequence can be generated from a unit measure and powers of the plastic number. Given a unit measure a paper folding technique can be used to construct ψ and powers of ψ. All Perrin numbers can be generated from the unit measure and the three constructible numbers ψ, $1/\psi$, and ψ^{-5}.***

Remember that the cubic equation $x^3 - x - 1 = 0$ has three solutions, one real and 2 complex numbers. Although imaginary solutions cannot be constructed with origami methods, in the case where the real solution is a morphic number the sum of powers of the complex number and its complex conjugate can be constructed.

Corollary***: Let ψ_1 and ψ_2 be the complex and complex conjugate solution to the equation $x^3 - x - 1 = 0$. All powers $\psi_1^n + \psi_2^n$ can be generated from the real numbers 2, $-\psi$ and ψ^{-5}.***

The Perrin sequence as shown in Chalkboard 1 is generated from powers of the three solutions to the cubic equation.

[4] **P(n)** = $\psi^n + \psi_1^n + \psi_2^n$

where n = 0,1,2,….

Let the powers of the complex solutions be constructed as follows:

$\psi_1^0 + \psi_2^0 = 2$

$\psi_1^1 + \psi_2^1 = -\psi$

$\psi_1^2 + \psi_2^2 = \psi^{-5}$

19.2 Theorem: Given the three generators for n = 0, 1, and 2 all powers are obtained from the recurrence relation $\psi_1^n + \psi_2^n = \psi_1^{n-2} + \psi_2^{n-2} + \psi_1^{n-3} + \psi_2^{n-3}$. The Perrin sequence P(n) is completely generated with positive and negative powers of ψ^n as in equation [4].

Use Theorems 1 and 2 to generate P(8) = 10.

From Theorem 1 we can show that:

$\psi^8 = 1+\psi +1 + 1/\psi + 1 + \psi + \psi + 1 + 1/\psi = 4*\mathbf{1} + 3*\boldsymbol{\psi} + 2*\mathbf{1/\psi} = 9.48390920$

From Theorem 2

$\psi_1^8 + \psi_2^8 = \psi^{-5} +2 - \psi +2 - \psi - \psi + \psi^{-5} = 2*\boldsymbol{\psi^{-5}} +2*\mathbf{2} - 3*\boldsymbol{\psi} = 0.51609080$

$\psi^8 + \psi_1^8 + \psi_2^8 = 9.48390920 + 0.51609080 = 10.00000000$

Corollary***: Let ψ be the real solution to the equation $x^3 - x - 1 = 0$. All powers ψ^n can be generated from the real numbers 1(unit), $1/\psi$ and ψ. The number of unit measures in the generation of ψ^n is the nth Padovan number.***

If all integer values are removed from the expansion of ψ^8 and $\psi_1{}^8 + \psi_2{}^8$ we find only two remaining irrational numbers in ψ; namely $2*1/\psi$ and $2*\psi^{-5}$. The sum of these two numbers is a unit measure $2*(1/\psi + \psi^{-5}) = 2*1$. Calculation of other powers of $\psi^n + \psi_1{}^n + \psi_2{}^n$ and subtracting the integers from the expansion shows that the remaining sums, $n*(1/\psi + \psi^{-5})$, increase as a Padovan sequence. This proves the following theorem:

19.3 Theorem*: The Padovan sequence is generated from the Perrin sequence after subtraction of the integer generators of powers of the real and complex solutions. The remaining metrics $1/\psi$ and ψ^{-5} sum to a unit measure. The number of these unit measures increases as the Padovan sequence.*

Since these metrics are constructible many interesting geometric figures can be built based on the Padovan sequence. For example, sides of triangles and rectangles of unit length can be built from $1/\psi + \psi^{-5}$ and volumes and areas calculated in units of ψ^2 . Several interesting relations involving these metrics can be useful:

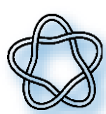
[5a] $2*1/\psi + \psi^{-5} = \psi^2$ [5b] $\psi^{-14} + 4*\psi^{-5} = 1$ [5c] $2*(\psi - \psi^{-1}) - \psi^{-7} = 1$

Let $f(x) = x^3 - x - 1 = 0$. A cubic equation is considered reciprocal if $x^3 * f(1/x) = x^3*(1/x^3 - 1/x - 1) = +/- f(x)$.

This cubic is non-reciprocal since $x^3 * f(1/x) = -x^3 - x^2 + 1 \neq +/- f(x)$. It can be shown that ψ satisfies the equation,

$$\psi^4 - \psi^3 - \psi^2 + 1 = 0$$

A larger degree polynomial is the Jones polynomial (S. Stahl, **Geometry from Euclid to Knots**, Dover Publications, Inc, 2010) for the "interlaced pentagram" or *cinquefoil* knot:



Jones Polynomial: $-q^7 + q^6 - q^5 + q^4 + q^2$

Although not an equation in x the Jones polynomial was developed as an invariant to distinguish whether two knots are equivalent. Two knots are different if their Jones polynomials are not equivalent.

It is interesting that ψ satisfies the polynomial $-\psi^7 + \psi^6 - \psi^5 + \psi^4 + \psi^2 + 1 = 0$ where the Jones polynomial for the "unknot" is 1.

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