The Perrin Conjugate and the Laguerre Orthogonal Polynomial

In a previous chapter I defined the conjugate of a cubic polynomial $G(x) = x^3 - Bx^2 - Cx - D$ as $G(x)_c = x^3 + Bx^2 - Cx + D$. By multiplying the polynomial with its' conjugate one obtains the first polynomial of order 2;

[1] $$P2(x,2) = x^6 + (B^2-2C)\, x^4 + (C^2-2BD)\, x^2 - D^2$$

The Perrin sequence is associated with solutions to $G(x) = x^3 - x - 1 = 0$. Its' Perrin conjugate, $G(x)_c = x^3 - x + 1$ multiplied by G(x) yields a degree six polynomial,

[2] $$P2\{x,2) = x^6 - 2\, x^4 + x^2 - 1$$

Successive multiplication of P2(x,2) with its conjugates gives the recursive formula,

[3] $$P2(x,n) * P2(x,n)_c = P2(x,2n).$$

In this chapter, I investigate another sequence which is related to the Perrin conjugate. Many well- known functions can be expanded as summations in powers of x. As an example, the exponential function e^x is expanded as,

[4] $$F(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \frac{x^{11}}{39916800} + \cdots ..$$

or in terms of a summation,

[5] $$F(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

In this case the denominator of the expansion is a series of factorials in n. Many other exponential functions can be expressed as an expansion of factorials. For example,

[6] $$F(x) = \mathrm{Sinh}[x] = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362880} + \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800} + \cdots ..$$

For some functions, the denominator of the expansion does not need to be a common factorial. An example I will discuss further is the expansion of $e^x * G(x)_c$, specifically when $G(x)_c = x^3 - x + 1$.

[7] $$F(x) = e^x * G(x)_c = 1 - \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{8} + \frac{7x^5}{15} + \frac{23x^6}{144} + \frac{17x^7}{420} + \frac{47x^8}{5760} + \frac{31x^9}{22680} + \frac{79x^{10}}{403200} + \frac{7x^{11}}{285120} + \cdots$$

In this case the denominator is not an obvious factorial and the numerator is an unknown series of coefficients. If these sequences are searched on OEIS then the numerator sequence 2,7,7,23,17,47.. is close to sequence A164314 described as the largest prime factor of $(n-1)^2-2$. For example for n = 10, $(10-1)^2-2 = 79$ which is a prime and the largest factor of itself. Unfortunately, as the number of terms increases the numerator is not the largest prime factor for all terms (e.g. at n = 29, $(29-1)^2-2 = 782 = 23*17*2$) but the factor is indicated as 1. The reason for these discrepancies will be discussed.

A search of the denominator sequence 2,3,8,15,144,420,5760… on OEIS is described as the value of the denominator of the Laguerre polynomial of degree n at x = 1. The expansion of $e^x * G(x)_c$ is then related indirectly to the Laguerre polynomial.

The Laguerre Polynomial

The Laguerre polynomial is one of several classical orthogonal polynomials found in mathematics. A standard source that I will be using is found in reference (1).

The closed form definition of the generalized Laguerre polynomial of degree n is

[8] $$L_n^{\alpha}(x) = \sum_{i=0}^{n}(-1)^i * \binom{n+\alpha}{n-i} * \frac{x^i}{i!}$$

Where $\binom{n+\alpha}{n-i}$ is the binomial coefficient equal to $(n+\alpha)!/((n-i)!\,(\alpha+i)!)$.

The Laguerre polynomial for $\alpha=0$ is defined as $L_n^0(x) = L_n(x)$.

The exponential generating function for $L_n(1)$ is

[9] $$e^{x/x-1}/(1-x) = 1 - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{5x^4}{8} - \frac{7x^5}{15} - \frac{37x^6}{144} - \frac{17x^7}{420} + \frac{887x^8}{5760} + \frac{1405x^9}{4536} + \frac{168919x^{10}}{403200} + \cdots ..$$

Comparing equation [9] with [7] notice that some denominators are different but are integer multiples. Although this disproves that the numerator of [7] is the largest prime factor of $(n-1)^2-2$ it strengths the hypothesis that the denominator is a denominator of $L_n(1)$.

The polynomial for a given n with $\alpha=0$ can be calculated from [8]. A cubic polynomial at n=3 is

[10] $$L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3) \quad \text{with} \quad L_3(1) = -\frac{2}{3}$$

agreeing with the 3rd coefficient term in [9]. Note that the coefficient for $L_1(1) = 0$

An exponential generating function can be used to find the expansion coefficients for the generalized Laguerre polynomial.

[11] $$L_n^{\alpha-1}(c) = e^{-\frac{cx}{1-x}}/(1-x)^{\alpha}$$

Where c is an integer and α is any rational number $\alpha \geq 0$ or $\alpha \leq 0$.

Equation [7] can also be generalized to give the magnitude of the denominator of $L_n^{\alpha-1}(c)$

[12] $$Denominator[L_n^{\alpha-1}(c)] = f * e^{cx} * (1 - x - x^3)^{\alpha}$$

where f is a positive integer generally equal to 1 and $\alpha \geq 0$ or $\alpha \leq 0$.

As with most orthogonal functions, any monomial term can be expressed as a sum of Laguerre polynomials. By adding these monomial terms together and multiplying by the associated coefficient any polynomial can be expressed as a sum of Laguerre polynomials.

As an example, $x^3 = -6L_3(x) + 18L_2(x) - 18L_1(x) + 6L_0(x)$, $x^1 = -L_1(x) + L_0(x)$ and

$1 = L_0(x)$ and the summation gives for the Perrin conjugate:

[13] $$x^3 - x + 1 = -6L_3(x) + 18L_2(x) - 17L_1(x) + 6L_0(x)$$

The value of the function on the LHS can then be verified for x=1 from associated Laguerre coefficients found in equation [9] or for any x = c from equation [11]

The orthogonality of the Laguerre polynomial can be useful for integrating polynomials formed from the summation of Laguerre polynomials. For α=0 Laguerre polynomials satisfy the condition,

[14] $$\int_0^\infty L_n(x) * e^{-x} * L_m(x)dx = \delta_{mn}$$

where δ_{mn} = 1 if n=m and 0 otherwise.

Multiplication of both sides of [13] by $L_n(x) * e^{-x}$ and integrating in the limit 0 to infinity results in a simplification of the integration of the LHS due to orthogonality on the RHS. In general,

[15] $$\int_0^\infty L_n(x) * e^{-x} * G(x)dx = a_n$$

Where a_n is the coefficient of $L_n(x)$ for expansion of G(x) with Laguerre polynomials.

The expansion coefficients a_n for each monomial can conveniently be found either in tables such as in reference (1) [Table 22.10], or by application of a Groebner basis. In *Mathematica,* expand the series in unknown coefficients of decreasing n using the Laguerre command and then use the Groebner basis command to solve for the coefficients. An example for obtaining x^3 is,

[16] In = $\text{Expand}[a * \text{LaguerreL}[3, x] + b * \text{LaguerreL}[2, x] + c * \text{LaguerreL}[1, x] + d * \text{LaguerreL}[0, x]]$

Out = $a + b + c + d - 3ax - 2bx - cx + \frac{3ax^2}{2} + \frac{bx^2}{2} - \frac{ax^3}{6}$

In = $\text{GroebnerBasis}[\{\frac{3a}{2} + \frac{b}{2}, -3a - 2b - c, a + b + c + d, a + 6\}, \{a, b, c, d\}]$

Out = $\{-6 + d, 18 + c, -18 + b, 6 + a\}$

If the expansion is made with the generalized Laguerre polynomial then the integration becomes

[17] $$\int_0^\infty L_n^\alpha(x) * e^{-x} * x^\alpha * G(x)dx = a_n * \Gamma(\text{n} + \alpha + 1)/\text{n}!$$

and the infinite range of integration on the LHS can be avoided by using the Gamma function of the RHS.

Expansion of Classic Orthogonal Polynomials with Generalized Laguerre Polynomials

Although a method such as [16] can be used for polynomials G(x) of any degree, as the degree increases the number of individual unknown coefficients to be solved increases as (n+1) (n+2)/2. For a polynomial of the 6th degree, 28 coefficients need to be found.

The problem posed in this section is: *Can an orthogonal polynomial be expanded in terms of the generalized Laguerre polynomials?* The answer is *yes* for the following polynomials; Legendre, Hermite, and Chebyshev. We seek a solution such that given an orthogonal polynomial $X_m(x)$, there exists an expansion;

[18] $$X_m(x) = \sum_{i=0}^m A_{m,i} * L_{m_i}^{\alpha_i}(0) * x^i$$

where $A_{m,i}$ is a coefficient dependent on m and i, and $L_{m_i}^{\alpha_i}(0)$ is the generalized Laguerre polynomial evaluated at x = 0 with coefficients m_i and α_i also dependent on i. I will use the notation of reference (1) where $P_m(x)$ is the Legendre polynomial, $H_m(x)$ the Hermite polynomial, $T_m(x)$ the Chebyshev T polynomial, $U_m(x)$ the Chebyshev U polynomial, $S_m(x)$ the Chebyshev S polynomial, and $C_m(x)$ the Chebyshev C polynomial.

Using the orthogonality property, once these expressions are found then the nth term $X_{mi}x^i$ of $P_m(x)$, $H_m(x)$, $T_m(x)$, $U_m(x)$, $S_m(x)$ or $C_m(x)$ can be found by integration with the appropriate generalized Laguerre polynomial as in [19] and the Gamma function.

[19] $$X_{mi}x^i = \frac{\int_0^{\infty} L_{b(m,i)}^{\alpha(m,i)}(z)*e^{-z}*z^{\alpha(m,i)}*c(m,i)*L_{b(m,i)}^{\alpha(m,i)}(z)*x^i dz}{\alpha(\mathrm{m,n})!} = \frac{\Gamma(\alpha(m,n)+b(m,n)+1)*c(m,n)*x^i}{b(m,n)!*\alpha(m,n)!}$$

where α(m,i), b(m,i) and c(m,i) are constant coefficients dependent on the degree m and the desired ith term of $X_m(x)$. In general, these integrals converge rapidly and integration to infinity is not required if performed numerically.

As an example, compute the coefficient for x^6 (I = 6) in the degree 12 Chebyshev S polynomial, $S_{12,6}x^6$. In equation [19] the 6th power of x is $x^{(m-2n)}$, n = 3 and the constants are α(12,6) = n = 3, b (12,6) = m-2n = 6 and c (12,6) = $(-1)^n$ = -1.

[20] $$S_{12,6}x^6 = \frac{\int_0^{\infty} L_6^3(z)*e^{-z}*z^3*(-1)*L_6^3(z)*x^6 dz}{3!} = \frac{\Gamma(6+3+1)*(-1)*x^6}{6!*3!} = -84x^6$$

The result agrees with Table 22.8 in reference (1) for matrix element A[S_{12}(x), x^6]. Since this table stops at S_{12}(x) any coefficient of S_{12}(x) can be determined from [20]. Note that this equation applies only for even values of m. Equations for even and odd values of m will be described for all orthogonal polynomials $X_m(x)$. Also, equation [20] converges to the value -84 to twenty decimal places for integration limits 0 to ≥100.

Associated with equation [19] which computes the individual coefficients, each orthogonal polynomial can be completely calculated using equation [18]. For the Chebyshev S polynomial, this expansion for m(even) is,

[21] $$S_m(x) = \sum_{n=0}^{m/2}(-1)^{\mathrm{n}} * L_{m-2n}^{n}(0) * x^{(m-2n)}$$

[22] $$S_{12}(x) = 1 - 21x^2 + 70x^4 - 84x^6 + 45x^8 - 11x^{10} + x^{12}$$

The following set of equations [23] to [34] are expansions of each orthogonal polynomial in terms of $L_{b(m,i)}^{\alpha(m,i)}(0)$. They have been tested in *Mathematica* using the appropriate commands for the generalized Laguerre polynomial. These derived expansions may not be the most optimal in form but are the basis for the integral equation [19]. These integral equations are unique since they can be used to calculate individual terms for other orthogonal polynomials. They are important in observing symmetry in the group of monomial coefficients of orthogonal polynomials; some of these symmetries will be described later.

Orthogonal Polynomial expansions: m(even)

[23] $$H_m(x) = \sum_{n=0}^{m/2}(-1)^{(m+2n)/2} * (2*(m-2n)! * 2^{2n-1}/((m-2n)/2)!) * \mathrm{L}_{2n}^{m-2n}(0) * x^{2n}$$

[24] $$U_m(x) = \sum_{n=0}^{m/2}(-1)^{\mathrm{n}} * \mathrm{L}_{m-2n}^{n}(0) * (2x)^{m-2n}$$

[25] $$S_m(x) = \sum_{n=0}^{m/2}(-1)^{\mathrm{n}} * \mathrm{L}_{m-2n}^{n}(0) * x^{m-2n}$$

[26] $$T_m(x) = \sum_{n=0}^{m/2}(-1)^{(m+2n)/2} * \left(\mathrm{L}_{2n}^{\frac{m}{2}-n}(0) + \mathrm{L}_{2n}^{\frac{m}{2}-n-1}(0)\right)/2 * (2x)^{2n}$$

[27] $C_m(x) = \sum_{n=0}^{m/2}(-1)^{(m+2n)/2} * \left(\mathrm{L}_{2n}^{\frac{m}{2}-n}(0) + \mathrm{L}_{2n}^{\frac{m}{2}-n-1}(0)\right) * x^{2n}$

[28] $P_m(x) = (de)\sum_{n=0}^{m/2}(-1)^{(m+2n)/2}\,(2 * ((m+2n)/2)!)/(((m-2n)/2)!\,(2n)!) * \frac{\mathrm{L}_{(m+2n)/2}^{\frac{(m+2n}{2}-1}(0)}{(2*\mathrm{fe})} * x^{2n}$

[28a] $\mathrm{fe} = \mathrm{Numerator}[2^{(m-1)}/(m)!]$

[28b] $\mathrm{de} = 1/\mathrm{Numerator}[2^m * m!^2/(2m)!]$

Orthogonal Polynomial expansions: m(odd)

[29] $H_m(x) = \sum_{n=0}^{(m-1)/2}(-1)^{(m+2n+3)/2} * (2 * (m-2n-1)! * 2^{2n}/((m-2n-1)/2)!) * \mathrm{L}_{2n+1}^{m-2n-1}(0) * x^{2n+1}$

[30] $U_m(x) = \sum_{n=0}^{(m-1)/2}(-1)^{\mathrm{n}} * \mathrm{L}_{m-2n}^{n}(0) * (2x)^{m-2n}$

[31] $S_m(x) = \sum_{n=0}^{(m-1)/2}(-1)^{\mathrm{n}} * \mathrm{L}_{m-2n}^{n}(0) * x^{m-2n}$

[32] $T_m(x) = \sum_{n=0}^{(m-1)/2}(-1)^{(m+2n-1)/2} * \left(\mathrm{L}_{2n+1}^{\frac{(m-1)}{2}-n}(0) + \mathrm{L}_{2n+1}^{\frac{(m-1)}{2}-n-1}(0)\right)/2 * (2x)^{2n+1}$

[33] $C_m(x) = \sum_{n=0}^{(m-1)/2}(-1)^{(m+2n-1)/2} * \left(\mathrm{L}_{2n+1}^{\frac{(m-1)}{2}-n}(0) + \mathrm{L}_{2n+1}^{\frac{(m-1)}{2}-n-1}(0)\right) * x^{2n+1}$

[34] $P_m(x) =$

$(do)\sum_{n=0}^{(m-1)/2}(-1)^{(m+2n-1)/2}\,(2 * ((m+2n+1)/2)!)/(((m-2n-1)/2)!\,(2n+1)!) * \frac{\mathrm{L}_{(m+2n+1)/2}^{\frac{(m+2n-1)}{2}}(0)}{(2*\mathrm{fo})} * x^{2n+1}$

[34a] $\mathrm{fo} = \mathrm{Numerator}[2^{(m-1)}/(m)!]$

[34b] $\mathrm{do} = 1/\mathrm{Numerator}[2^m * m!^2/(2m)!]$

Integral Representations of Orthogonal Polynomial Monomial terms.

$$X_{mi}x^i = \frac{\int_0^\infty L_{b(m,n)}^{\alpha(m,n)}(z) * e^{-z} * z^{\alpha(m,n)} * c(m,n) * L_{b(m,i)}^{\alpha(m,i)}(z) * x^i dz}{\alpha(\mathrm{m,n})!} = \frac{\Gamma(\alpha(m,n) + b(m,n) + 1) * c(m,n) * x^i}{b(m,n)! * \alpha(m,n)!}$$

[35] Hermite Polynomial $H_{mi}x^i$,m even

$$n = i/2$$

$$c(m,n) = (-1)^{(m+2n)/2} * (2 * (m-2n)! * 2^{2n-1}/((m-2n)/2)!)$$

$$\alpha(m,n) = m - 2n$$

$$b(m,n) = 2n$$

$$(\Gamma[a+b+1]/b!) * (-1)^{((m+b)/2)} * 2^b/\,(a/2)! * x^b$$

[36] Hermite Polynomial $H_{mi}x^i$,m odd

$$n = (i-1)/2$$

$$c(m,n) = (-1)^{(m+2n+3)/2} * (2*(m-2n-1)! * 2^{2n}/((m-2n-1)/2)!)$$

$$a(m,n) = m-2n-1$$

$$b(m,n) = 2n+1$$

$$(\Gamma[a+b+1]/b!) * (-1)^{(a)/2} * 2^b/(a/2)! * x^b$$

[37] Legendre Polynomial $P_{mi}x^i$,m even

$$n = i/2$$

$$c(m,n) = (-1)^{(m+2n)/2} * (2*((m+2n)/2)!)/(((m-2n)/2)!\,(2n)!)\,/(2*\mathrm{fe})$$

$$a(m,n) = (m+2n)/2-1$$

$$b(m,n) = (m+2n)/2$$

$$\mathrm{fe} = \mathrm{Numerator}[2^{(m-1)}/m!]$$

$$(-1)^b * (\Gamma[2b])/\big(((m-i)/2)!\,(i)! * (\mathrm{fe}) * a!\big) * x^i$$

[38] Legendre Polynomial $P_{mi}x^i$,m odd

$$n = (i-1)/2$$

$$c(m,n) = (-1)^{(m-1+2n)/2} * (2*((m+1+2n)/2)!)/(((m-1-2n)/2)!\,((2n+1))!)\,/(2*\mathrm{fo})$$

$$a(m,n) = (m-1+2n)/2$$

$$b(m,n) = (m+1+2n)/2$$

$$\mathrm{fo} = [\mathrm{Numerator}[2^{(m-1)}/m!]$$

$$(-1)^a * (\Gamma[2b])/\big(((m-i)/2)!\,(i)! * (\mathrm{fo}) * a!\big) * x^i$$

[39] Chebychev S Polynomial $S_{mi}x^i$,m even

$$n = (m-i)/2$$

$$c(m,n) = (-1)^n$$

$$a(m,n) = n$$

$$b(m,n) = m-2n$$

$$(\Gamma[a+b+1]/b!) * (-1)^a * x^b/a!$$

[40] Chebychev S Polynomial $S_{mi}x^i$,m odd

$$n = (m - i)/2$$

$$c(m,n) = (-1)^n$$

$$\alpha(m,n) = n$$

$$b(m,n) = m - 2n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^a * x^b/a!$$

[41] Chebychev U Polynomial $U_{mi}x^i$,m even

$$n = (m - i)/2$$

$$c(m,n) = (-1)^n * 2^{(m-2n)}$$

$$\alpha(m,n) = n$$

$$b(m,n) = m - 2n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^a * 2^b * x^b/a!$$

[42] Chebychev U Polynomial $U_{mi}x^i$,m odd

$$n = (m - i)/2$$

$$c(m,n) = (-1)^n * 2^{(m-2n)}$$

$$\alpha(m,n) = n$$

$$b(m,n) = m - 2n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^a * 2^b * x^b/a!$$

The following Chebyshev polynomials consist of two integrations

$$X_{mi}x^i = \frac{\int_0^\infty L_{b(m,n)}^{\alpha(m,n)}(z) * e^{-z} * z^{\alpha(m,n)} * c(m,n) * L_{b(m,i)}^{\alpha(m,i)}(z) * x^i dz}{\alpha(m,n)!} + \frac{\int_0^\infty L_{b(m,n)}^{\alpha 1(m,n)}(z) * e^{-z} * z^{\alpha 1(m,n)} * c(m,n) * L_{b(m,i)}^{\alpha 1(m,i)}(z) * x^i dz}{\alpha 1(m,n)!}$$

$$X_{mi}x^i = \frac{\Gamma(\alpha(m,n)+b(m,n)+1)*c(m,n)*x^i}{b(m,n)!*\alpha(m,n)!} + \frac{\Gamma(\alpha1(m,n)+b(m,n)+1)*c(m,n)*x^i}{b(m,n)!*\alpha1(m,n)!}$$

[43] Chebychev T Polynomial $T_{mi}x^i$,m even

$$n = i/2$$

$$c(m,n) = (-1)^{(m+2n)} * 2^{2n}/2$$

$$\alpha(m,n) = m/2 - n - 1$$

$$b(m,n) = 2n$$

$$\alpha1(m,n) = m/2 - n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^{((m+b)/2)} * 2^{(b-1)} * x^b/a! + (\Gamma[a1 + b + 1]/b!) * (-1)^{((m+b)/2)} * 2^{(b-1)} * x^b/\text{a1}!$$

[44] Chebychev T Polynomial $T_{mi}x^i$,m odd

$$n = (i-1)/2$$

$$c(m,n) = (-1)^{(m+2n-1)} * 2^{2n+1}/2$$

$$\alpha(m,n) = (m-1)/2 - n - 1$$

$$b(m,n) = 2n + 1$$

$$\alpha1(m,n) = (m-1)/2 - n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^{((m+b-2)/2)} * 2^{(b-1)} * x^b/a! + (\Gamma[a1 + b + 1]/b!) * (-1)^{((m+b-2)/2)} * 2^{(b-1)} * x^b/\text{a1}!$$

[45] Chebychev C Polynomial $C_{mi}x^i$,m even

$$n = i/2$$

$$c(m,n) = (-1)^{(m+2n)/2}$$

$$\alpha(m,n) = m/2 - n - 1$$

$$b(m,n) = 2n$$

$$\alpha1(m,n) = m/2 - n$$

$$(\Gamma[a + b + 1]/b!) * (-1)^{((m+b)/2)} * 2^{(b)} * x^b/a! + (\Gamma[a1 + b + 1]/b!) * (-1)^{((m+b)/2)} * 2^{(b)} * x^b/\text{a1}!$$

[46] Chebychev C Polynomial $C_{mi}x^i$,m odd

$$n = (i-1)/2$$

$$c(m,n) = (-1)^{(m+2n-1)/2}$$

$$a(m,n) = (m-1)/2 - n - 1$$

$$b(m,n) = 2n$$

$$a1(m,n) = (m-1)/2 - n$$

$$(\Gamma[a+b+1]/b!) * (-1)^{((m+b-1)/2)} * 2^{(b)} * x^b/a! + (\Gamma[a1+b+1]/b!) * (-1)^{((m+b-1)/2)} * 2^{(b)} * x^b/\text{a1}!$$

The following computations are easily done with Mathematica. Integration limits are typically 10*m. Calculation with the Gamma function is faster.

Compute $P_{23,17}x^{17}$ (use integration limit 0 to 10*23)

$$\text{fo} = [\text{Numerator}[2^{(23-1)}/23!] = 8 \qquad \text{n = (17-1)/2 = 8}$$

$$a(m,n) = \left(23 - 1 + 2(8)\right)/2 = 19$$
$$b(m,n) = (23 + 1 + 2(8))/2 = 20$$

$$c(m,n) = (-1)^{19} * \frac{(2*(20)!)/(3)!(17)!}{2*8} = -2280/16$$

$$P_{23,17}x^{17} = \frac{\int_0^{10*m} L_{20}^{19}(z) * e^{-z} * z^{19} * (-1) * \frac{2280}{16} * L_{20}^{19}(z) * x^{17} dz}{19!} = -\frac{39!}{6 * 17! * 8 * 19!}$$
$$= -9821565178425x^{17}$$

This number agrees with the expansion for $P_{23}(x)$ found in equation [34]

$$P_{23}(x) = -2028117x + 185910725x^3 - 5019589575x^5 + 62386327575x^7 - 429772478850x^9 + 1805044411170x^{11} - 4859734953150x^{13} + 8562390155550x^{15} - 9821565178425x^{17} + 7064634602025x^{19} - 2893136075115x^{21} + 514589420475x^{23}$$

Equations [43] and [44] and [45] and [46] demonstrate that

$$T_{mi}x^i = 1/2(U_{mi}x^i - U_{(m-2)i}x^i)$$

$$C_{mi}x^i = (S_{mi}x^i - S_{(m-2)i}x^i)$$

This symmetry is also listed in [22.5.8] and [22.5.12] of reference (1).

Monomial Term Symmetry

Using the above integral representations, knowing conditions for symmetry in the generalized Laguerre polynomial leads to symmetry of one or more of the orthogonal polynomials. As a simple example, consider the Legendre polynomial $P_{m,i}x^i$. For any m, if i = m then given m1 = 2m and i = 0 the coefficients for the generalized Laguerre polynomial are equal and $P_{m,m}x^m = -P_{m,0}x^0$. Also, looking at the Hermite polynomial $H_{m,m}x^m = 2^k H_{m-k,m-k}x^{m-k}$ where $L^0_{m,m}$ and $L^0_{m-k,m-k}$ differ by 2^k.

For the Legendre polynomial, the relation $P_{m,m-2}x^{m-2} = \frac{f(m)}{f(2(m-2)+1)} * P_{2(m-2)+1,1}x^1$ depends on the values of f(m) = fo(m) or fe(m) with m even or odd. The Legendre polynomial equations [37] and [38] are suitable for finding several symmetries. For example, by partitioning (a+b) in [37 and [38] into 3 parts the following relation is found,

$$P_{m,i}x^i = \frac{f(m/2 + 3i/2)}{f(m)} * P_{\frac{m}{2}+\frac{3i}{2},(m-i)/2}x^{(m-i)/2}$$

Unfortunately, the generalized Laguerre polynomial has no known recurrences relating $\alpha(m,n)$ and $b(m,n)$ to another different set $\alpha(m,n)$ and $b(m,n)$ with $\alpha(m,n)$>0. However, the Laguerre polynomial is directly related to a confluent hypergeometric function, the Kummer function M[a, b, z] (see reference (2), equation 10.12.14]. A direct substitution of the Kummer polynomial function for the Laguerre polynomial in the expansion of the Hermite polynomial [23] yields;

$$H_m(x) = \sum_{n=0}^{m/2} (-1)^{(m+2n)/2} * (2 * (m-2n)! * 2^{2n-1}/((m-2n)/2)!) * \left(\frac{m!}{(m-2n)! * (2n)!} * \mathrm{M}[-2\mathrm{n}, \mathrm{m}-2\mathrm{n}+1, 0]\right) * x^{2n}$$

The integral representation of the monomials for the Hermite polynomial in terms of the Kummer function from [35] is:

$$H_{mi}x^i = (\int_0^\infty L^{\alpha(m,n)}_{b(m,n)}(z) * e^{-z} * z^{\alpha(m,n)} * c(m,n) * \left(\frac{m!}{\alpha(m,n)! * b(m,n)!} * \mathrm{M}[-b(m,n), \alpha(m,n)+1, \mathrm{z}]\right) * x^i dz)/\alpha(\mathrm{m},\mathrm{n})!$$

where the Kummer term in parenthesis can also replace $L^{\alpha(m,n)}_{b(m,n)}(z)$. Further research in this area is required to discover hidden symmetries using other representations of the confluent hypergeometric functions.

The above equations, [35] to [46] illustrate a direct connection of the orthogonal polynomials to symmetric functions. The term $(\Gamma[a+b+1]/a!\,b!) = (a+b,b)!$ is the integer coefficient of the term $x_1^a x_2^b$ in the expansion of $(x_1+x_2)^{(a+b)}$. For the Chebyshev S and U polynomials,

*The x^b term of the Chebyshev polynomial X_{mb} is c(m,n) *[coefficient of $x_1^a x_2^b$] in the expansion of $(x_1+x_2)^{(a+b)}$.*

A similar connection can be made for the Hermite and Legendre polynomials:

*The x^b term of the Hermite polynomial H_{mb} is c(m,n) *[coefficient of $x_1^a x_2^b$] in the expansion of $(x_1+x_2)^{(a+b)}$.*

*The x^i term of the Legendre polynomial P_{mi} is c(m,n) *[coefficient of $x_1^a x_2^b$] in the expansion of $(x_1+x_2)^{(a+b)}$.*

These functions are symmetric since the coefficients $x_1^a x_2^b = x_1^b x_2^a$. The relation of univariate orthogonal polynomials to bivariate symmetric polynomials has many applications in algebraic number theory particularly in symmetric group representation and combinatorial algorithms. See reference (3).

This study began with investigation of equation [7]. The sequence in the denominator is found equivalent to the exponential generating function for the Laguerre polynomial evaluated at unity [9]. The Perrin conjugate equation multiplies the exponential and shifts the factorial denominator in powers of x. For this reason, the denominator of [7] follows a series 1/n! -1/(n-1)! + 1/(n-3)!. This result is equivalent to the nth term found in [7]. Although equation

[7] did not lead to any further mathematical consequence, the concept lead to the use of the generalized Laguerre polynomial in the expansion of other classical orthogonal polynomials as demonstrated in equations [23] to [46]. It is hoped that these expansions will be a useful reference to those using classical orthogonal polynomials.

(1) **Handbook of Mathematical Functions**, National Bureau of Standards Applied Mathematics Series 55, M. Abramowitz and I. Stegun (Ed), (1964) pp 771-802.

(2) **Higher Transcendental Functions, Vol. II,** Bateman Manuscript Project, H. Bateman, (1953), pp 178-196.

(3) **The Symmetric Group [Ed. 2]**, B. Sagan, Springer, (2000).

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