The Jacobi Polynomial, Laguerre Polynomial and Delannoy Number

In Chapter 23 we found Laguerre expansions of various orthogonal polynomials. Since the Laguerre polynomial L(m, a, x) is also orthogonal, these expansions could be integrated over the positive x axis with weight e^{-x} * x^a to give the coefficient of each term of degree x^n. This was done for various Chebyshev polynomials, the Hermite polynomial and the Legendre polynomial.

The Jacobi Polynomial P(m, a, b, x) is a general orthogonal form of the Legendre Polynomial P(m, x). For integers a = 0 and b= 0, P(m, 0, 0, x) = P(m, x). The Jacobi polynomial is given as an expansion in terms of $(x - 1)^n$ so the complete form in x^n requires an expansion in powers of $(x - 1)^n$. It is convenient to start with

[1] $$\mathrm{P(m,a,b,x)} = (\mathrm{Gamma}[a+m+1]/(\mathrm{Gamma}[a+b+m+1]*m!))*$$

$$\sum_{n=0}^{m}\mathrm{Binomial}[m,n]* \ \mathrm{Gamma}[a+b+m+n+1]/\mathrm{Gamma}[a+n+1]*(x-1)/2^n$$

From the definition of the Laguerre Polynomial it is simple to show that this can be written as:

[2] $$\mathrm{P(m,a,\ b,x)} = (\mathrm{Gamma}[a+m+1]/(\mathrm{Gamma}[a+b+m+1]*m!))*$$

$$\sum_{n=0}^{m}\mathrm{Binomial}[m,n]*\mathrm{L}[b+m,a+n,0]*(b+m)!*(x-1)/2^n$$

Integrating the term in the summation of [2] *wrt* variable z and L(b+m, a+n, z),

[3] $$\int_0^\infty e^{-z}z^{a+n}\mathrm{Binomial}[m,n](b+m)!\,\mathrm{L}[b+m,a+n,z]^2 2^{-n}(x-1)^n\,dz$$

we obtain,

[4] $$\mathrm{Gamma}[b+m+a+n+1]*\mathrm{Binomial}[m,n]/((a+n)!)*2^{-n}(x-1)^n$$

When the terms preceding the summation in [2] are included, the nth coefficient in $(x - 1)^n$ is

[5] $$(\mathrm{Gamma}[a+m+1]/(\mathrm{Gamma}[a+b+m+1]*m!))*\mathrm{Gamma}[b+m+a+n+1]$$

$$*\,\mathrm{Binomial}[m,n]/((a+n)!)*2^{-n}(x-1)^n$$

Since the Legendre and Jacobi polynomials are usually given as coefficients in x^n, it is necessary to expand and collect all terms $(x - 1)^n$. This is best done by the summation of [2];

[6] $$\mathrm{P(m,a,b,x)} = \sum_{n=0}^{m}(\mathrm{Gamma}[a+m+1]/(\mathrm{Gamma}[a+b+m+1]*m!))\mathrm{Binomial}[m,n]*$$

$$\mathrm{L}[b+m,a+n,0]*(b+m)!*2^{-n}(x-1)^n$$

In *Mathematica*, the function [6] can be directly expanded to yield terms in x^n.

It is interesting to compare equations [2] to [6] with the equations obtained for the Legendre polynomial in Chapter 23. It can be shown that equation [28] and [34] in that Chapter give the same result as equation [6] for P(m, 0, 0, x) as expected. However, since the maximum degree of P(m, 0, 0, x) is m, there is only one x^m term to collect in [6]. In equation [5] let a = 0, b = 0 and n = m. Then the final term x^m, in [6] is given simply by,

[7] $$\mathrm{Gamma}[2m+1]/(m!\,m!)*2^{-m}$$

Unexpectedly, the denominator of this term are the constants de and do defined previously.

[8] $$\text{de} = 1/\text{Numerator}[(2^m * m!^2/(2m)!]$$

and removing the Numerator designation,

[9] $$\text{de}' = \text{Gamma}[2m+1]/(m!\,m!) * 2^{-m}$$

Also, the terms fe and fo previously described for computing the Legendre polynomial were found in OEIS A048896. By simplifying the terms for x^m in equation [28] and [34] of the previous Chapter, the following equations are derived,

[10a, b] $$\text{fe} = (((2m-1)! * (-1)^m/((m-1)! * m!))/\text{de}') * \text{de}$$

$$\text{fo} = (((2m-1)! * (-1)^{m-1}/((m-1)! * m!))/\text{de}') * \text{de}$$

The orthogonality of P(m, a, b, x) can be proved by integrating [2] *wrt* z with L(b + m', a + n, z) where m' ≠ m for all terms in n,

[11] $$\int_0^\infty e^{-z} z^{a+n} \text{Binomial}[m,n](b+m)!\,\text{L}[b+m',a+n,z]^* \text{L}[b+m,a+n,z] 2^{-n}(x-1)^n\,dz$$

After expanding equation [6] standardized orthogonal polynomials are obtained with [12] as standardization,

[12] $$h_n = 2^{a+b+1}/(2m+a+b+1) * \text{Gamma}[m+a+1] * \text{Gamma}[m+b+1]/(m! * \text{Gamma}[m+a+b+1])$$

such that

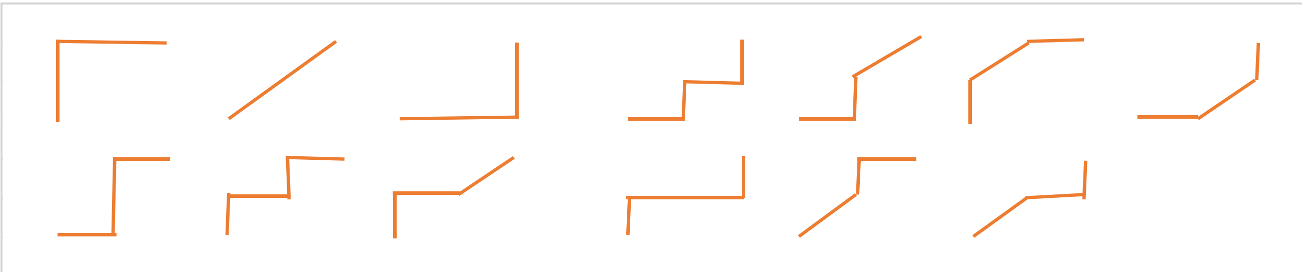
[13] $$\int_{-1}^1 \text{P}(\text{m},\text{a},\text{b},\text{z}) * (1-z)^a (1+z)^b * \text{P}(\text{m}',\text{a},\text{b},\text{z})\,dz = h_n$$

if m = m' and zero, otherwise.

Also, equations [1] and [2] are solvable with positive and negative integers and half fraction values for parameters a and b.

The Delannoy Number

The Delannoy number D(m, n) is an integer sequence which quantifies the number of walks on a lattice plane from point (0, 0) to point (m, n). The constraint is that only walks (0, 1), (1, 0) and (1, 1) are allowed and they can only occur in a north, east or northeast direction. Given a central point (m, m) on a square lattice the value of D(m, m) = P(m,3) the value of the Legendre polynomial at x = 3. For example D(2, 2) = P(2, 3) = 13. An illustration of these walks from (0, 0) to (2, 2) is shown:



There is no known combinatoric correspondence of the Legendre Polynomial to the Delannoy numbers (see below Delannoy numbers revisited). However, Gabor Hetyei (reference (1) demonstrated that the asymmetric Delannoy number $D^{\sim}(m,n)$ could be expressed as a Jacobi Polynomial P(n, 0, m-n, 3). $D^{\sim}(m,n)$ is the number of walks on a lattice plane from (0, 0) to (m, n + 1) where each step (x,y) is a nonnegative integer in x and a positive integer in y. Expressing a = 0, b = m – n and x = 3 in equation [5] the following equation for the asymmetric Delannoy number is derived: (note: the variable m is exchanged with n and the original n is replaced with the summation variable k)

[14] $$D^{\sim}(m,n) = \sum_{k=0}^{n}(n!/((b+n)!)) * (b+n+k)!/((n-k)! * k!^{2})$$

As an example of [14] with n = 3 and m= 2, and b = m - n = -1, $D^{\sim}(m,n)$ = 38. (see Table of Asymmetric Delannoy Numbers in reference (1)).

A more extensive table read as diagonals of the table in reference (1) is given in OEIS A049600. The (m+n)th diagonal is calculated by the following equation using Laguerre polynomials derived from [2];

[15] $$D^{\sim}[\mathrm{n'\_, n\_}] := \sum_{i=0}^{k}[\mathrm{L}[n'-n,i,0] * \mathrm{L}[n-i,i,0]$$

To use this equation n' = m + n and n = n. From the example above n' = 5 and n = 3 giving the same result $D^{\sim}[\mathrm{n'\_, n\_}]$ = 38. (See OEIS A049600 for more detail).

Equation [15] also has a connection to n-simplices. An n-simplex is an n dimensional triangle. Starting with a triangle 2-simplex of 3 vertices adding a point orthogonal to the other 3 points represents a tetrahedron. The tetrahedron has 4 vertices, 6 edges, 4, 2-faces and 1, 3- face which is the whole 3-simplex tetrahedron. Adding another point to the 3-simplex results in a 4-simplex or 5-cell of 5 vertices, 10 edges, 10, 2 faces, 5, 3 faces and 1, 4- face.

It can be shown that if i = (i - 1)th-face, and n' = (n - 1)-Simplex then integrating [15] with the Laguerre polynomial $\mathrm{L}[n'-n,i,z]$ yields,

[16] $$\int_0^\infty \mathrm{L}[n'-n,i,z] * e^{-z}e^{-z}z^{i} * \frac{\mathrm{L}[n'-n,i,z]}{i!} * \mathrm{L}[n-i,i,0]\,dz = (n)!/((n-i)! * i!) * \mathrm{Binomial}[n'-n+i,i]$$

Further setting n = i results in a simple binomial equation Binomia$[n'$, i] for the number of (i-1)-faces for the n' = (n-1)-Simplex. For example, n' = 8 represents the 7-Simplex and i = 5 is the 4-face then;

[17] Number of 4-faces of a 7-simplex = $\mathrm{Binomial}[8,5]$ = 56

The f-vector for an n-simplex is the series of faces {f_{-1}, f_0, f_1, ….., $f_{(n-1)}$} where f_{-1}=1 and the series represents the number of points, edges, 2-face, 3-face etc. The associated f- polynomial uses this vector as coefficients,

[18] $f(x) = f_{(-1),} *(x - 1)^n + f_0 *(x - 1)^{n-1} + f_1 * (x - 1)^{n-2} +........+ f_{(n-1)}$

The expansion of [18] results in the corresponding polynomial h(x) with coefficients for powers of x.

[19] $h(x) = h_{(-1),} *(x)^n + h_0 *(x)^{n-1} + h_1 * (x)^{n-2} +........+ h_{(n-1)}$

where the h vector is $\{h_0, h_1, ...h_{(n-1)}\}$ calculated by equation [20],

[20] $$h_i = \sum_{j=0}^{i} (-1)^{i-j} * \text{Binomial}[n-j, i-j] * f_{j-1}$$

For all of the n=simplices, the h vector is a series of n, 1's; {1, 1, 1, 1, ……}

Hetyei discusses a 2-color interpretation of the asymmetric Delannoy number and its relation to the Jacobi and Legendre polynomial. He proves that $D^{\sim}(m,n)$ enumerates all 2-colored lattice paths from points (0, 0) to (m, n), m and n are nonnegative integers, under the following rules:

(i) Each step is either a north direction blue (0, 1) or a red (x, y) with x and y nonnegative integers not both zero.
(ii) At least one of any two consecutive steps is blue (0, 1).

He shows that the number of 2- colored lattice paths is;

[21] $$D^{\sim}(m,n) = \sum_{j=0}^{n} Binomial[n,j] * Binomial[m+j,j]$$

where j is the number of blue steps. From [21] we see that the number of blue steps ranges from 0 to n. The steps must have a partial order such that (p,q) < (p', q') iff p ≤ p' and q < q'. As an example, consider paths from (0, 0) to (1, 3). If we call the number of blue steps the "partial chains" then the number of partial chains containing 0, 1, 2, and 3 blue steps is calculated from [21] is (1, 6, 9, 4). The sum is $D^{\sim}(m,n) = 20$. The first chain has no blue but only a red chain from (0, 0) to (x, y) = (1,3). To identify the remaining chains, define the following 6-line segments: a = (0, 0) to (0, 1), b = (0, 1) to (0, 2), c = (0, 2) to (0,3), d = (1, 0) to (1, 1), e = (1, 1) to (1, 2) and f = (1, 2) to (1, 3). The 6-one chains are then (a, b, c, d, e, f) the 9-two chains are (ab, bc, a and c, de, ef, d and f, a and e, a and f, b and f) and the 4-three chains (ab and f, a and ef, abc, def). The lattice paths are then completed by inserting red steps in the remaining spaces.

Hetyei compares these partial chains to simplicial complexes which are a set of vertices and line segments (the blue steps). Comparing the partial chains to the f -vector, this f- vector can be expanded in powers of (x-1)/2 to yield a corresponding Jacobi polynomial. In the example above $\{f_{-1}, f_0, f_1, f_2\}$ = {1, 6, 9, 4). Then we find,

[22] $$P(n, 0, m-n, x) = \sum_{j=0}^{n} f_{j-1} * ((x-1)/2)^j = -\frac{1}{4} + \frac{3}{4} * x^2 + \frac{1}{2} * x^3$$

Applying equation [22] with x = 3 gives the correct result $D^{\sim}(m,n) = 20$.

If we dissect equation [21] into the individual paths for a given j, each term is the product $Binomial[n,j] * Binomial[m+j,j]$. Comparing to equation [16] above the first binomial is the number of (j-1) faces for the (n-1) simplex. Each term in the f-vector is then a (j-1)-face of the (n-1) simplex times a binomial which depends on j and m. The above example of the f vector $\{f_{-1}, f_0, f_1, f_2\}$ = {1, 6, 9, 4) is generated from [21] with n= 3, m= 1 and j= 0 …n. This can be designated as an (n-1) simplex vector (1, s_0, s_1,..s_n) representing a 2-simplex (triangle) with {j-1} = (-1, 0, 1, 2) terms; 1, 3-vertices, 3 edges and 1-triangle (the (-1) term is always 1). Each j-1 term is multiplied by a "property" vector (1, p_0, p_1.., p_n), $Binomial[m+j,j]$, for the (j-1)-face.

*Conjecture 1: The asymmetric Delannoy number D~ (m,n) is the dot product of the (n-1) simplex vector **s** with the (j-1) property vector **p**.*

Example: Let (m,n) = (2, 9). The 8-simplex has the s vector {1,9,36,84,126,126,84,36,9,1}. The $Binomial[m+j,j]$ term can also be equivalently expressed as $Binomial[m+j,m]$. For m = 2 the p vector is {1,3,6,10,15,21,28,36,45,55}. The vector product results in the sum of terms in the f-vector {1, 27, 216, 840, 1890, 2646, 2352, 1296, 405, 55} = 9728. Looking at the s_0 = 9 vertices of the 8-simplex if we append or join each of the p_0 = 3 distinct 'properties' to each vertex we obtain 27 distinct possible outcomes. To accomplish this requires making 3 copies of the 0-face of the 8-simplex and attaching one distinct property to each of the nine vertices. The process is repeated for each (j-1) face of the 8-simplex until the 8-simplex is replicated 55 times and each receives a distinct property of p_9.

The property term can be interpreted as the number of distinct colorings of k objects with c colors using all colors at least once. The binomial provides an exact interpretation by letting m+1 = number of colors and m+1+j be the number of objects to be colored. In the example j = 1 ((j-1) is the 0-face) so we have 4 objects and 3 colors. Let a, b, c, represent the three colors then there are 3 distinct colorings abca, abcb, and abcc. Apply each coloring to copies of the 9 – 0 faces. Note that the size of the objects increases by one as j increases, until j = 9 where the appended property consists of 12 objects with 55 distinct colorings using 3 colors.

The asymmetric Delannoy number is then the total number of these complexes appended to multiple copies of the (n-1) simplex. The total number of copies is the sum of the terms in the p-vector and each copy has a unique property.

Hetyei extends his theory to simplicial complexes of higher dimension and shows that the asymmetric Delannoy number counts the number of partial chains in a "balanced (all vertices colored differently) join" of simplicial complexes with an (n-1) dimensional colored simplex. Details can be found in reference (2).

The property vector can also be defined from the symmetry group, S_m. Given (m,n), the (n-1) simplex vector is defined by the faces of the (n-1) simplex. The property vector is generated from the cycle index of the symmetry group S_m. The cycle index polynomial for S_2 is $\frac{x_1^2}{2}+\frac{x_2}{2}$. If x_1 and x_2 are indexed as the value of m+j-1 (j = 0 …. n) the values of the cycle index polynomial are {1,3,6,10,15,21,28,36,45,55} for n = 9 as above. The symmetry group S_m is used as the property vector for all values of m to calculate the asymmetric Delannoy number. It would be interesting to define numbers using a different finite group G to define this property such as the cyclic group or the dihedral group. The interpretation of the resulting number is left as an exercise.

The Delannoy number revisited

Contrary to the literature, the Delannoy number D(m,n) has a close connection to the Jacobi polynomial beyond the central diagonal. The following connection follows the same proof as above. Expressing b = 0, a = m – n and x = 3 in equation [5] the following equation for the Delannoy number is derived: (note: the variable m is exchanged with n and the original n is replaced with the summation variable k):

[23] $$D(m,n) = \sum_{k=0}^{n} (a+n+k)!/(k!*(n-k)!*(a+k)!)$$

As an example of [23] with n = 3 and m= 2, and a = m - n = -1, $D(m,n)$ = 25. (see Table of Delannoy Numbers in reference (1)).

The Delannoy number is also calculated by the following equation using Laguerre polynomials derived from [2] (compare with [15] above;

[24] $$D(m,n) := \sum_{i=0}^{n} [\mathrm{L}[m-n+i,n,0] * \mathrm{L}[n-i,i,0]]$$

It is not surprising that Equation [24] also has a connection to n-simplices. Equation [24] can be written in binomial form:

[25] $$D(m,n) = \sum_{j=0}^{n} Binomial[n,j] * Binomial[m+j,n]$$

Note the slight difference in the second binomial term from equation [21]. As expected there is a similar correlation of the Delannoy number to the (n-1) simplex and a property vector as we found for the asymmetric Delannoy number. The following conjecture defines this correlation.

*Conjecture 2: The Delannoy number D (m,n) is the dot product of the (n-1) simplex vector **s** with an n-property vector **p**.*

Based on the previous analysis for the asymmetric Delannoy number the most convenient property vector is from a **shifted** cycle index polynomial for the symmetric group S_n. The shift is positive for m > n, negative for m < n, and no shift for m = n.

Examples: Calculate D(m,n) = D(3,6). The (n-1) simplex vector s of the f-1 = 1, and 0- face to the 5-face is {1, 6, 15, 20, 15, 6, 1}. Consecutive values for the cycle index polynomial of the symmetric group S_6 are 1, 7, 28, 84, 210, 482, …. The p vector is shifted left by 3-6 = -3 units by replacement with zeros. The remaining n+1-3 = 4 places in the vector are filled as {0, 0, 0, 1, 7, 28, 84}. The dot product of these vectors is 377. For the reverse, D(m,n) = (6,3) the corresponding 2-simplex vector is multiplied by a right shifted by 3 vector of S_3. {1, 3, 3, 1} . {20, 35, 56, 84} = 377. This example demonstrates a symmetry in the Delannoy number: D(m,n) = D(n,m). For the central numbers, the dot product of a (n-1) simplex vector with un-shifted values for the cycle index polynomial of the symmetric group $S_{n.}$ results in the value of the n-Legendre polynomial in x evaluated at x = 3.

Conjecture 3: The Delannoy number D(m,n) is the dot product of the n-th row of Pascal's triangle with the m-th extended diagonal of the coefficients for the Chebyshev U polynomial.

Chebyshev U polynomials were developed from an expansion of the Laguerre polynomial in Chapter 23. It was shown in equation [42] that the monomials of the Chebyshev U polynomial can be expressed by:

[26] $$\frac{\Gamma(a+b+1)}{b!} * (-1)^a * 2^b * \frac{x^b}{a!}$$

Removing the sign and the variable x the up-diagonal coefficients can be expressed as:

[27] $$Binomial[b+a,b] * 2^b$$

with **b** increasing from 0 to m and **a** decreasing as m-b. The m-th diagonal has m+1 terms but is extended by adding zeros to the end of the vector. For example. When m = 4 the first 5 terms are extended with zeros as {1, 8, 24, 32, 16, 0, 0, 0, 0, 0, 0…). The 4-th row D(4, n) is then developed by taking the dot product with the n-th row of Pascal's triangle {the (n-1) simplices are a subset of the Pascal triangle}. Then D(4, 7) = {1, 7, 21, 35, 35, 21, 7, 1}.{{1, 8, 24, 32, 16, 0, 0,0} = P(4, 7-4, 0, 3) = 2241.

OEIS A001846 describes the D(4, n) sequence is described as the "crystal ball sequence for a 4-dimensional cubic lattice". Looking at other sequences, D(3, n) is OEIS A001845, the "crystal ball sequence for a cubic (3-dimensional) lattice" or centered octahedral numbers. D(2, n) is OEIS A001844, the "crystal ball sequence for a square (2-dimensional) lattice". D(5, n) is OEIS A001847, the "crystal ball sequence for a 5-dimensional cubic lattice". OEIS (reference 3) defines the crystal ball sequence as the n-th term giving the number of vertices which are at most n edge traversals away for a given vertex on a vertex transitive graph [of dimension m].

These results confirm that there is an underlying topological significance of the Delannoy and asymmetric Delannoy numbers. Both numbers are also expressible with Jacobi polynomials, Laguerre polynomials and binomials.

1. G. Hetyei, Links we almost missed between Delannoy numbers and Legendre polynomials
2. G. Hetyei, Central Delannoy numbers, Legendre polynomials, and a balanced join operation preserving the Cohen- Macaulay property, *Formal Power Series and Algebraic Combinatorics*, Series Formelles et Combinatoire Algebrique, San Diego, CA., 2006.
3. J. H. Conway, N. J. A. Sloane, Low-Dimensional Lattices. VII. Coordination sequences, *Proceedings of the Royal Society*, 453:1966 (1997), pp. 2369-2389.

Richard Turk

September 16, 2017