The Electrical Potential Laplace Operator

In this Chapter we seek a solution to Laplace's equation using orthogonal functions. We consider a membrane exposed to an electrical field. The membrane is a thin sheet of infinite length and width on the x, z plane and of thickness 2h in the y direction. A field is created in the y direction by applying a voltage to cathode and anode electrodes of length 2L along the x axis, where L >> h. Since the electrodes are of finite length, a region of the membrane outside the electrode region experiences a field strength not equal to zero. This region outside the electrode region is also considered in the solution to the problem. We assume this outside membrane boundary is non-conductive and the electric field perpendicular to the boundary is zero. Under conditions of electroneutrality, no free charge is assumed within the membrane and Laplace's equation applies to the problem.

A model of the problem shows the upper quadrant y> 0, in the x, y plane of the membrane (see figure 1). The solutions are symmetric about the upper and lower quadrants. In this chapter solutions are developed for $0 \geq x > -\infty$ and $h \geq y \geq 0$.

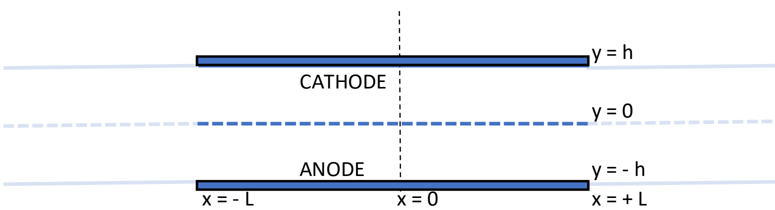

Figure 1_ A membrane electrode system. Within region |x|< L an electrical field is applied. The boundary region outside the electrodes is insulated.

Since the boundary conditions are mixed around the electrodes, the method of solution will be with a decomposition of the partial differential equation into two domains. The operator will be made self adjoint and orthogonal eigenfunctions are found inside and outside the electrode regions. The inside regions are defined for $0 \geq x \geq -L$ and $h \geq y \geq 0$ and the outside region $-L > x > -\infty$ and $h \geq y \geq 0$.

Statement of Problem

Let $\phi(\xi, \eta) = V(x, y)/V_o$ be the dimensionless potential at coordinates $(\xi, \eta) = (x/h, y/h)$ and V_o the potential applied to the cathode. The Laplace potential equation in a plane is:

[1] $$\frac{\partial^2 \emptyset}{\partial \zeta^2} + \frac{\partial^2 \emptyset}{\partial \eta^2} = 0$$

We assume that the center line $\eta = 0$, is at zero potential. Within the electrode region at $\eta = 1$ the potential is constant. Outside the electrode region the boundary is an insulated/ non-conductor and the perpendicular electric field is zero.

[2a, b, c]
$$\phi(\xi, 0) = 0 \quad \text{all } \xi$$
$$\phi(\xi, 1) = 1 \quad |\xi| \leq L$$
$$\frac{d\phi(\zeta,1)}{d\eta} = 0 \quad |\xi| > L$$

From symmetry requirements around the center of the electrode at $\xi = 0$,

[3a, b]

$$\frac{d\phi(0,\eta)}{d\zeta} = 0$$

$$\frac{d\phi(0,\eta)}{d\eta} = 1$$

These conditions indicate that at the center of the electrode region the field is in a perpendicular direction to the centerline.

Conditions at the electrode

The electrode creates a discontinuity at $\xi = +/- L$ where the applied potential drops from one to zero. Also, the electric field perpendicular to the electrode, $\frac{d\phi(\zeta,\eta)}{d\eta}$, suddenly changes from a constant value to zero as the boundary condition is crossed. This suggests that the field will become infinite as ξ approaches +/- L at $\eta = 1$. To overcome this discontinuity, define a function $f(\xi)$ at the electrode boundary such that,

[4]

$$\frac{d\phi(\zeta,1)}{d\eta} = f(\xi) + 1$$

with $|\xi| \leq L$. The function $f(\xi)$ is symmetric around $\xi = 0$ such that $f(\xi) = 0$ at $\xi = 0$.
Since the outside boundary is an insulator one expects polarization at this boundary as the value of $f(\xi)$ approaches infinity. The polarization creates an electrical field in the ξ direction which extends from the upper to the lower boundary outside the electrode region. In the development of a solution, the infinity at $(+/-L, 1)$ and $(+/-L, -1)$ can be controlled through the expansion of the eigenfunctions and modeled as the magnitude of polarization at this interface.

Conversion of PDE to set of ODEs

Let $Z(\xi,\eta) = \phi(\xi, \eta) - \eta$ where Z satisfies Laplace's equation [1] with boundary conditions:

[5a, b, c]

$$Z(\xi, 0) = 0 \quad \text{all } \xi$$

$$Z(\xi, 1) = 0 \quad |\xi| \leq L$$

$$\frac{dZ(\zeta,1)}{d\eta} = \{ -1 \quad |\xi| > L$$

$$= \{ f(\xi) \quad |\xi| \leq L$$

Define the function Σ as;

[6]

$$\Sigma(\zeta,\eta) = -\int_0^{\eta} \frac{dZ}{d\zeta} d\eta$$

Differentiate [6] wrt ξ to obtain the first decomposition:

[7]

$$d\Sigma(\zeta,\eta)/d\zeta = -\int_0^{\eta} \frac{d}{d\zeta}\frac{dZ}{d\zeta} d\eta = \int_0^{\eta} \frac{d}{d\eta}\frac{dZ}{d\eta} d\eta = \frac{dZ(\zeta,\eta)}{d\eta} - \frac{dZ(\zeta,0)}{d\eta} = dZ(\zeta,\eta)/d\,\eta$$

Differentiate [6] wrt η to obtain the second decomposition:

[8] $$d\Sigma(\zeta,\eta)/d\eta = -\int_0^{\eta} \frac{d}{d\zeta}\frac{dZ}{d\eta}\, d\eta = -\int_0^{\eta} \frac{d}{d\eta}\frac{dZ}{d\zeta}\, d\eta = -\frac{dZ(\zeta,\eta)}{d\zeta} + \frac{dZ(\zeta,0)}{d\zeta} = -dZ(\zeta,\eta)/d\,\zeta$$

From [7] and the boundary conditions for $\frac{dZ(\zeta,\eta)}{d\eta}$ at the electrode [5c] we can integrate over each region:

[9a, b, c]
$$\int_{\Sigma(\zeta,1)}^{\Sigma(-L,1)} d\Sigma(\zeta,\eta) = -\int_{\zeta}^{-L} d\zeta = \quad \Sigma(-L,1) - \Sigma(\zeta,1) = L + \zeta$$
$$\int_{\Sigma(L,1)}^{\Sigma(\zeta,1)} d\Sigma(\zeta,\eta) = -\int_{L}^{\zeta} d\zeta = \quad \Sigma(\zeta,1) - \Sigma(L,1) = L - \zeta$$
$$\int_{\Sigma(-L,1)}^{\Sigma(\zeta,1)} d\Sigma(\zeta,\eta) = -\int_{-L}^{\zeta} f(\zeta)\, d\zeta = \quad \Sigma(\zeta,1) - \Sigma(-L,1) = I(\zeta)$$

where [9a, b] are outside the electrode and [9c] inside the electrode. Rearranging [9a, b, c] we can solve for the term $\Sigma(\zeta,1)$ at the upper boundary.

[10a, b, c]
$$\Sigma(\zeta,1) = \Sigma(-L,1) - L + |\zeta| \qquad \text{for } \xi < -L$$
$$\Sigma(\zeta,1) = \Sigma(L,1) + L - |\zeta| \qquad \text{for } \xi > L$$
$$\Sigma(\zeta,1) = \Sigma(-L,1) + I(\zeta) \qquad \text{for } |\xi| \le L$$

The function $I(\zeta)$ and the Eigenfunctions

In [9c] the integral of electrode function is defined as

[11] $$I(\zeta) = \int_{-L}^{\zeta} f(\zeta)\, d\zeta$$

The value of this function can be derived at ξ = 0. Since $\frac{dZ(0,\eta)}{d\zeta} = 0$

[12]
$$\Sigma(0,1) = \Sigma(-L,1) + I(0) = 0 \text{ for } |\xi| \le L$$
$$I(0) = \int_{-L}^{0} f(\zeta)\, d\zeta$$

Define a new function :

[13] $$W(\zeta,\eta) = \Sigma(\zeta,\eta) + \gamma\,(I(\zeta) + L - |\zeta|)$$

where γ is -1 if $\zeta < 0$ and 1 if $\zeta \ge 0$. It can be shown that W(ζ, 1) satisfy the boundary conditions,

[14] $$W(\zeta,1) = \gamma\,(I(\zeta) + L - |\zeta|) \qquad \text{for } |\xi| \le L$$

$$W(\zeta, 1) = 0 \qquad \text{for } |\xi| > L$$

Differentiating [13] wrt |ξ| the equations [7], [8] and [13] are written in vector (transpose) notation as,

[15]
$$\frac{d\Psi(\zeta,\eta)}{d\zeta} = \mathcal{M}\Psi + G$$

where $\Psi = (Z, W)^T$, $G = (0, 1)^T$ and $\mathcal{M}$ is the two dimensional operator $\begin{bmatrix} 0 & -d/d\eta \\ d/d\eta & 0 \end{bmatrix}$.

The operator can be shown to be self-adjoint in both domains of Hilbert space $\mathfrak{H}$.

Domain 1(outside electrode) of $\mathcal{M} = \{\phi \in \mathfrak{H}\text{: } \mathcal{M}\phi \text{ exists}, \tau(0) = 0 \text{ and } \sigma(1) = 0\}$
Domain 2(inside electrode) of $\mathcal{M} = \{\phi \in \mathfrak{H}\text{: } \mathcal{M}\phi \text{ exists}, \tau(0) = 0 \text{ and } \sigma(1) = 0\}$

The eigenfunctions and eigenvalues in each domain obey $\mathcal{M}\phi = \lambda\phi$ and are summarized in the Table below:

Domain	Applied to	Eigenfunctions $\phi_i = (\tau_i,\sigma_i)$	Eigenvalues $\lambda_{(i)}$
1	Outside electrode	$\sin(\lambda_{(i)}*\eta)$, $\cos(\lambda_{(i)}*\eta)$	$\lambda_{(i)} = (2i + 1) * \pi/2$
2	Inside electrode	$\sin(\lambda 1_{(i)}*\eta)$, $\cos(\lambda 1_{(i)}*\eta)$	$\lambda 1_{(i)} = (i + 1) * \pi$

The inner product of two vectors in this space is defined by the integration;

[16]
$$d<\Psi, \phi_i>/d\xi = <\mathcal{M}\,\Psi, \phi_i>/ + <G, \phi_i>$$

with, in general for any vector of functions $u_1 = <t_1, \sigma_1>$ and $u_2 = <t_2, \sigma_2>$

[17]
$$< u_1, u_2> = \int_0^1 \tau_1\,\tau_1 + \sigma_1\sigma_1\,d\eta$$

Where the norm $<\phi_i|\phi_i> = 1$.

We seek solutions to the potential function Z(ξ,η) of [15] as an expansion of the inner product with the eigenfunctions $\phi_\iota(\eta)$. The solution of [15] also includes an exponential expansion in ξ. Looking at Domain 1, the solution requires expansion in both positive and negative eigenvalues over an infinite range of ξ.

[18a, b]
$$<\Psi, \phi^+_i> = C^+e^{\lambda^+\zeta} + \tau_j{}^+(1)\int_\zeta^\infty W(\zeta',1)e^{\lambda^+(\zeta-\zeta')}d\zeta' - \int_\zeta^\infty < G,\ \phi^+ > e^{\lambda^+(\zeta-\zeta')}d\zeta'$$
$$<\Psi, \phi^-_i> = C^-e^{\lambda^-\zeta} - \tau_j{}^-(1)\int_{-\infty}^\zeta W(\zeta',1)e^{\lambda^-(\zeta-\zeta')}d\zeta' - \int_{-\infty}^\zeta < G,\ \phi^- > e^{\lambda^-(\zeta-\zeta')}d\zeta'$$

For the solutions to be finite at +/- infinity, the constants $C^+ = C^- = 0$. The solution is expanded as

[19] $$Z(\xi,\eta) = \sum_{i=0}^{\infty} < \Psi, \Phi_i > * \Phi_i(\lambda_i \eta)$$

Since integrals are evaluated over the whole range of ξ, the first integral in [18a] is zero over the intervals |ξ| > L but is not zero in the interval |ξ| ≤ L due to [14]. This integral involves the function I(ξ) in [11].

[20] $$\int_{-L}^{L} W(\zeta', 1) e^{\lambda^+(\zeta-\zeta')} d\zeta' = \int_{-L}^{L} \gamma \left(I(\zeta) + L - |\zeta|\right) \; e^{\lambda^+(\zeta-\zeta')} d\zeta$$

Let K = $\int_{-L}^{L} I(\zeta') \; e^{-\lambda^+(\zeta')} d\zeta'$. Evaluating the integral in [20] I get,

[21] $$\int_{-L}^{L} \gamma \left(I(\zeta) + L - |\zeta|\right) \; e^{\lambda^+(\zeta-\zeta')} d\zeta = e^{\lambda^+(\zeta')} (K - \frac{2}{(\lambda^+)^2} + \frac{2}{(\lambda^+)^2} * \cosh(\lambda^+ L))$$

The remaining integral is

[22] $$-\int_{\zeta}^{\infty} < G, \; \phi^+ > e^{\lambda^+(\zeta-\zeta')} d\zeta' = -\frac{\sin(\lambda^+)}{(\lambda^+)^2}$$

Plugging into [19],

[23] $$Z(\xi,\eta) = \sum_{i=0}^{\infty} [e^{\lambda_i{}^+(\zeta')} \sin(\lambda_i{}^+) \left(K - \frac{2}{(\lambda_i{}^+)^2} + \frac{2}{(\lambda_i{}^+)^2} * \cosh(\lambda_i{}^+ L)\right) - \frac{\sin(\lambda_i{}^+)}{(\lambda_i{}^+)^2}] \sin(\lambda_i{}^+ \eta)$$
$$+ \sum_{i=0}^{\infty} (-\frac{\sin(\lambda_i{}^-)}{(\lambda_i{}^-)^2}) \sin(\lambda_i{}^- \eta)$$

Since $\sin(\lambda^+) = -\sin(\lambda^-)$ and $\eta = \sum_{i=0}^{\infty} \frac{\sin(\lambda^+)}{(\lambda^+)^2}] \sin(\lambda^+ \eta)$ with $Z(\xi,\eta) = \phi(\xi, \eta) - \eta$, the solution for $\phi(\xi, \eta)$ in domain 1 is:

[24] $$\phi(\xi,\eta) = \sum_{i=0}^{\infty} e^{(\lambda_i{}^+ \zeta)} \left(K_i - \frac{2}{(\lambda_i{}^+)^2} + \frac{2}{(\lambda_i{}^+)^2} * \cosh(\lambda_i{}^+ L)\right) \sin(\lambda_i{}^+) \sin(\lambda_i{}^+ \eta)$$

Repeating the above analysis for the solution inside the electrode at |ξ| ≤ L, I obtain:

[25] $$\phi(\xi,\eta) = \eta - \sum_{i=0}^{\infty} 2\beta_i e^{(-\lambda 1_i{}^+ L)} \cosh(\lambda 1_i{}^+ \zeta) \cos(\lambda 1_i{}^+) \sin(\lambda 1_i{}^+ \eta)$$

Here $$\beta_j = \sum_{i=0}^{\infty} \left(K_i - \frac{2}{(\lambda_i{}^+)^2} + \frac{2}{(\lambda_i{}^+)^2} * \cosh(\lambda_i{}^+ L)\right) * \frac{\lambda 1_j{}^+}{(\lambda_i{}^+ + \lambda 1_j{}^+)} * e^{(-\lambda_i{}^+ L)} - \frac{2}{(\lambda_i{}^+)^2}$$

Using this solution, we can solve for K_i in [23] (note that it is a generated from the eigenvalues λ_i). From [5c] the function f(ξ) is found from the boundary condition on $\frac{dZ(\zeta,\eta)}{d\eta}$ in domain 2. Let β_j be constant (O(1)) at the electrode ($\eta = 1$) then from [25] :

[26a, b, c]]

$$f(\xi) = \sum_{i=0}^{\infty} 2e^{(-\lambda 1_i{}^{+} L)} \cosh(\lambda 1_i{}^{+} \zeta)$$

$$I(\xi) = \int_{-L}^{\infty} (\sum_{i=0}^{\infty} 2e^{(-\lambda 1_i{}^{+} L)} \cosh(\lambda 1_i{}^{+} \zeta))\, d\zeta$$

$$K_i = \int_{-L}^{L} I(\zeta)\ e^{-\lambda^{+}(\zeta)} d\zeta = \frac{2Le^{(-\lambda_i{}^{+} L)}}{\lambda_i{}^{+}} - \frac{e^{(-\lambda_i{}^{+} L)}}{\lambda_i{}^{+2}} * \sinh(2\lambda_i{}^{+} L) + 4\frac{e^{(-\lambda_i{}^{+} L)}}{\lambda_i{}^{+2}} * [\sinh(2\lambda_i{}^{+} L)]^2$$

We find that [26a] for f(ξ) is infinite at +/- L, and is zero a $\xi = 0$. This amount of 'charge' at the electrode boundary at +/- L is made finite by summing to a finite value of i. Substitution of [26c] into [24] and [25] yields the complete solution to the mixed boundary problem for the 2D membrane with confined field. Evaluating the derivative of [24] wrt η shows that boundary condition [2c] is satisfied for η = 1. The discontinuity in the electric field at $\xi = -$L is most observed as η approaches unity and depends on the direction of approach. Conditions [2a] and [2b] are satisfied with the solution [25]. Other factors such as polarization at the insulated boundary and continuity at the electrode boundary would need to be considered if representing the true field behavior at $\xi = +/-$ L.
This method of solution can be applied to other 2-dimensional second order PDE's found in fluid flow and heat conduction problems.

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