The Jacobi Polynomial Revisited

Discoveries can be fortuitous if we look in the right places. Look back at Chapter 24 and discover some of the properties of the Jacobi Polynomial. We saw how the combinatoric Delannoy numbers can be written as Jacobi Polynomials evaluated at x = 3. We also showed how these values can be evaluated from vector dot products of rows of the Pascal triangle with either the appropriate cycle index of a symmetry group polynomial or an appropriate diagonal of the Chebyshev U polynomials.

I give a proof in this chapter that all Jacobi Polynomials P[m, a, b, x] evaluated at x=3 can be reduced to a series of Jacobi Polynomials of form P[n,c,1,3] with n= 0, 1, 2, … m. Each polynomial of the series is a vector dot or scalar product of rows of the Pascal triangle with an appropriate diagonal of the **Chebyshev T** polynomial. Any scalar product with the Chebyshev T polynomial can be reduced to a scalar product with a shifted cycle index polynomial of a symmetric group. The result proves that the scalar value of all Jacobi polynomials P[m, a, b, 3] can be transformed to scalar products.

Some Preliminaries

Three examples of the scalar product used to calculate P[m, 0, b, 3] and P[m, a, 0, 3] are reviewed from Chapter 24. Rows of the Pascal triangle are used in all three calculations.

Let m = 7, a=0, and b=5. Write P[7, 0, 5, 3] as P[7, 0, 12-7, 3]. Then using the $(m\text{-}1)^{th}$ row of Pascal, and the cycle index of the symmetry group S_{12}, apply the scalar product to obtain;

[1] $$\{1,7,21,35,35,21,7,1\}.\{1,13,91,455,1820,6188,18564,50388\} = 391912$$

Let m = 7, a=5, and b=0. Write P[7, 5, 0, 3] as P[7, 12-7, 0, 3]. Then using the $(m+a\text{-}1)^{th}$ row of Pascal, and the diagonal of the Chebyshev U polynomial for x^m, apply the scalar product as described in Chapter 24 to obtain;

[2] $$\{1,12,66,220,495,792,924,792,495,220,66,12,1\}.\{1,14,84,280,560,672,448,128,0,0,0,0,0\} = 1392065$$

Note that the i^{th} entry of the diagonal is given by the binomial equation [27] of Chapter 24. As the diagonal only has 7 terms the remaining elements of the vector are assigned zero value.

It is also possible to write P[7, 5, 0, 3] as P[7, 7-2, 0, 3] and use the shifted cycle index of the symmetry group S_7. Using the 6^{th} row of the Pascal triangle and shifting the cycle index 5 units to the right:

[3] $$\{1,7,21,35,35,21,7,1\}.\{792,1716,3432,6435,11440,19448,31824,50388\} = 1392065$$

Not that the Legendre polynomial P[7,3] with a= b= 0 can be calculated when the cycle index of S_7 is not shifted.

[3a] $$\{1,7,21,35,35,21,7,1\}.\{1,8,36,120,330,792,1716,3432\} = 48639$$

The above examples illustrate that any Legendre or Jacobi polynomial can be calculated if either or both a and b are zero. Below I discuss the general Jacobi Polynomial with a and b not equal to zero and consider whether a scalar product can be used to calculate P[m, a, b, 3][a] .

Reduction of the Jacobi Polynomial

I define the reduction of a general Jacobi polynomial as a transform of the value of the polynomial |P[m, a, b, 3]| to a summation of the form,

[4] $$|\mathrm{P}[m,a,b,3]| = \left|\sum_{i=0}^{i=m-1} c_i * \mathrm{P}[m-i, a+i, 1{,}3]\right| + c_m$$

This formula is valid only for the scalar value of the polynomial at x = 3 and is not applicable to all values of x. As shown below it is only valid at x = 3 and for real and complex values to a difference polynomial of degree m-1.

Several identities lead to equation [4]. Consider a simple polynomial of the form P[2, 2, i, 3] with i any positive integer. Let I = 4, then it can be shown that the value of the polynomial at x = 3 is;

[5] $$P[2{,}2{,}4{,}3] = \mathrm{P}[2{,}2{,}1{,}3] + \sum_{i=11}^{i=13} i$$

where the summations are based on the sums m+a+i+3*2-1. This equation only applies for m = 2 and a = 2. However, it leads to an interesting and important identity. Evaluate P[3,1,4,3]:

[6] $$\mathrm{P}[3{,}1{,}4{,}3] = \mathrm{P}[3{,}1{,}1{,}3] + \sum_{i=2}^{i=4} \mathrm{P}[2{,}2, i, 3]$$

Using [5] the summation is expanded as

[7] $$\mathrm{P}[3{,}1{,}1{,}3] + 3 * \mathrm{P}[2{,}2{,}1{,}3] + \sum_{i=11}^{i=11} i + \sum_{i=11}^{i=12} i + \sum_{i=11}^{i=13} i$$

The summations can be combined using OEIS A101860 to give the value of P[3, 1, i, 3] for any i;

[8] $$\mathrm{P}[3{,}1{,}1{,}3] + (i-1) * [2{,}2{,}1{,}3] + (3 + (i-2)) * (2 + 33(i-2) + (i-2)\text{^}2)/6 + 10$$

A similar formula can be used to evaluate P[3, 2, i, 3];

[9] $$\mathrm{P}[3{,}2{,}1{,}3] + (i-1) * \mathrm{P}[2{,}3{,}1{,}3] + (3 + (i-2)) * (2 + 33(i-2) + (i-2)\text{^}2)/6 + (i-1) * (i) + 10$$

Let P[m, a, i, 3] be a general Jacobi polynomial with x = 3. Then the following identities are true.

[10a] $$P[m,a,i,3] = P[m,a,1{,}3] + \sum_{i=2}^{i=m} P[m-1, a+1, i, 3]$$

[10b] $$P[m,a,b,3] = \sum_{i=0}^{i=m} P[i, m+a-i, b-1{,}3] = \sum_{i=0}^{i=m} P[m-i, a+i, b-1{,}3]$$

[10c] $$|\mathrm{P}[m,a,b,3]| = |\mathrm{P}[m,a,1{,}3] + \sum_{i=2}^{i=b} \mathrm{P}[m-1, a+1, i, 3]\,|$$

Using identities [10a] to [10c] it is possible to reduce any Jacobi polynomial at x = 3 to a form shown in equation [4].

Example: Reduce P[4,3,3,3] (value is 4809) to $\left|\sum_{i=0}^{i=4-1} c_i * \mathrm{P}[4-i, 3+i, 1{,}3]\right| + c_4$

Using [10c] and then [10a] ;

[11] $$\mathrm{P}[4{,}3{,}3{,}3] = \mathrm{P}[4{,}3{,}1{,}3] + \sum_{i=2}^{i=3} \mathrm{P}[3{,}4, i, 3]$$

[11b] $$= \mathrm{P}[4{,}3{,}1{,}3] + \mathrm{P}[3{,}4{,}2{,}3] + \sum_{i=3}^{i=3} \mathrm{P}[3{,}4, i, 3]$$

[11c] $$= \mathrm{P}[4{,}3{,}1{,}3] + \mathrm{P}[3{,}4{,}1{,}3] + \mathrm{P}[2{,}5{,}2{,}3] + \sum_{i=3}^{i=3} \mathrm{P}[3{,}4, i, 3]$$

[11d] $= P[4,3,1,3] + P[3,4,1,3] + P[2,5,1,3] + P[1,6,2,3] + \sum_{i=3}^{i=3} P[3,4,i,3]$

[11e] $= P[4,3,1,3] + P[3,4,1,3] + P[2,5,1,3] + P[1,6,1,3] + P[0,7,1,3] + \sum_{i=3}^{i=3} P[3,4,i,3]$

[11f] $= \sum_{i=0}^{i=4} P[4-i, 3+i, 1,3] + P[3,4,3,3]$

Reduce P[3,4,3,3] starting again with [10c] ;

[12a] $P[3,4,3,3] = P[3,4,1,3] + \sum_{i=2}^{i=3} P[2,5,i,3]$

[12b] $= P[3,4,1,3] + P[3,4,2,3] + \sum_{i=3}^{i=3} P[2,5,i,3]$

[12c] $= P[3,4,1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 1,3] + P[2,5,3,3]$

[12d] $= P[3,4,1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 2,3]$

[12e] $= P[3,4,1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 1,3] + P[2,5,2,3] + P[1,6,2,3] + P[0,7,2,3]$

[12f] $= P[3,4,1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 1,3] + \sum_{i=0}^{i=2} P[2-i, 5+i, 1,3] + \sum_{i=0}^{i=1} P[1-i, 6+i, 1,3] + \sum_{i=0}^{i=0} P[0-i, 7+i, 1,3]$

Combining [11f] and [12f], collecting similar terms and noting that P[0, a, 1, 3] = 1 the reduced form is;

[13] $P[4,3,3,3] = P[4,3,1,3] + 2 * P[3,4,1,3] + 3 * P[2,5,1,3] + 4 * P[1,6,1,3] + 5 = \sum_{i=0}^{i=3} c_i * P[4-i, 3+i, 1,3] + c_m$

Computer programming of the steps allows the lengthy calculations to be done efficiently.

Comparing the Jacobi polynomial with the reduced polynomial we find that they do not represent the same polynomial although they have the same degree m.

[14a] $P[3,4,3,x] = \frac{7}{16}(3 - 66x^2 + 143x^4)$

[14b] Reduced polynomial(x) = $\frac{3}{16}(79 + 198x + 300x^2 + 330x^3 + 165x^4)$

Factoring the difference of the two polynomial we find the difference to be zero at x = 3 as expected;

[15] $P[3,4,3,x] - reduced\ polynomial(x) = \frac{1}{8}(-3 + x)(36 + 111x + 264x^2 + 253x^3)$

If x is a solution to the degree 3 polynomial in [15] we find that $|P[3,4,3,x]| = |reduced\ polynomial\ (x)|$ for the real and complex solutions x = 2.54091, -0.16792 +/- 0.41579i.

Since equations [10] reduces m and b by 1, starting with any Jacobi polynomial there will be an eventual reduction to m = 0 an b=1. From the above analysis we can state:

Theorem 1: For any Jacobi Polynomial P[m,a,b,3] there exists a reduced polynomial of equal degree with all terms expressed as P[n,c,1,3] where n ranges from 0 to m and c increases from a to a + m.

Theorem 2: For any Jacobi Polynomial P[m,a,b,x] there is a reduced polynomial of equal degree (all terms expressed as P[n,c,1,x] that is equal to the Jacobi polynomial at x=3 and the difference is factored into terms (x-3) and a polynomial of degree m-1.

Scalar Products of the Jacobi Polynomial

We now turn to the purpose of converting the Jacobi polynomial to reduced form. In equations [1] to [3] above the Jacobi polynomial can be expressed as a scalar product of two vectors, one vector from a row of the Pascal triangle and the other vector obtained either from the cycle index of a symmetry group or with a diagonal of the Chebyshev U polynomial. These products are found only if either or parameters a and b are zero. It is also possible to express the reduced Jacobi polynomial P[n, c, 1, 3] as a scalar product of a row from the Pascal triangle and a diagonal from the Chebyshev T polynomial. This polynomial was introduced in Chapter 23 and again in Chapter 26.

Definition 1: *The reduced polynomial P[n, c, 1, 3] when expressed as P[n, n+c-n, 1, 3] has a value expressed by the (n-1)th row of the Pascal triangle and the n+c +1 diagonal of the Chebyshev T polynomial.*

Here $\{1,1\},\{1,2,1\},\{1,3,3,1\},\{1,4,6,4,1\}$ … …. are rows 0, 1, 2, 3, 4…… and as an example P[4, 3, 1, 3]= P[4, 7-4, 1, 3] is expressed as the dot product of the 3rd row and the Pascal triangle and the 8th diagonal of the Chebyshev T polynomial. $\{1,4,6,4,1\}.\{1,15,98,364,840\} = 2945$. In Chapter 23 equations [43] and [44] express the monomial terms of the Chebyshev T polynomial. It can be shown that the jth term (where 1 is the 0th term) of the t = n+c+1 diagonal is

[16] $$(\text{Binomial}[\text{t}, j] + \text{Binomial}[\text{t}-1, j]) * 2\char`^(j-1)$$

In the reduced form the parameter n+c = a for all terms. The tth Chebyshev T diagonal then applies to all terms and only reduced in length by the value of n. As an example, the reduced form in equation [13] is expressed by the scalar products;

[17] $$\text{P}[4,3,3,3] = \{1,4,6,4,1\}.\{1,15,98,364,840\} + 2 * \{1,3,3,1\}.\{1,15,98,364\} + 3 * \{1,2,1\}.\{1,15,98\} + 4 * \{1,1\}.\{1,15\} + 5 = 4809$$

For any fixed values of m and a, the reduced form requires m rows of the Pascal triangle, 1 diagonal of the Chebyshev T polynomial and m integer constants. The values of the integer constants only change values as the parameter b is increased or decreased. Here again we find a pattern associated with the Pascal triangle:

Definition 2*- The ith coefficient of the reduced scalar product form is calculated from the* $Binomial[n-2+i, n-2]$ *with i = 0, 1, 2, 3, 4 ,…..(b-m+1).*

Example: Expand the Jacobi Polynomial P[7,5,n,3] in reduced scalar product form.

[18]
$$\{1,7,21,35,35,21,7,1\}.\{1,25,288,2024,9680,33264,84480,160512\} + \text{Binomial}[n-1, n-2] * \{1,6,15,20,15,6,1\}.\{1,25,288,2024,9680,33264,84480\} + \text{Binomial}[n, n-2] * \{1,5,10,10,5,1\}.\{1,25,288,2024,9680,33264\} + \text{Binomial}[n+1, n-2] * \{1,4,6,4,1\}.\{1,25,288,2024,9680\} + \text{Binomial}[n+2, n-2] * \{1,3,3,1\}.\{1,25,288,2024\} + \text{Binomial}[n+3, n-2] * \{1,2,1\}.\{1,25,288\} + \text{Binomial}[n+4, n-2] * \{1,1\}.\{1,25\} + \text{Binomial}[n+5, n-2]$$

This calculation demonstrates that the scalar product of a Jacobi polynomial can be formed without knowing the reduced form!

Reduction to a Scalar Product with the cycle index of a Symmetry group

We next show how to express the reduced form [4] using the shifted cycle index of a symmetry group. Using identity [10b] with b = 1;

[19] $$\mathrm{P}[m,a,1,3] = \mathrm{P}[m,a,0,3] + \sum_{i=1}^{i=m} \mathrm{P}[m-i,a+i,0,3]$$

From the form of the scalar product in [18] it is possible to deduce the reduced form of each term. The pascal row is given by the second element in the vector which is m, and the second element of the Chebyshev T diagonal minus 1 is 2*(m+a). The first term in [18] is then P[7,5,1,3]. Using [19]

[20] $$\mathrm{P}[7,5,1,3] = \mathrm{P}[7,5,0,3] + \sum_{i=1}^{i=7} \mathrm{P}[7-i,5+i,0,3]$$

Unfortunately, this adds m new terms for each term of a reduced form. We can now use the shifted cycle index to calculate each term. As in the example shown in [3] above P[7,7-2,0,3] is calculated using the 6th row of the Pascal triangle and the cycle index of S_7 shifted by 5 units to the right. The corresponding terms in the summation of [20] are calculated by using the row number of the Pascal triangle for 7-i and the cycle index of the symmetry group S_{7-I} shifted by a+i.

Theorem 3: For all Jacobi Polynomials P[m ,a, b, 3] there exists a reduced polynomial of equal degree with all terms expressible as P[n_i,c_i,0,3]. Each term can be represented by a scalar product of the n_i th row of the Pascal triangle and the cycle index of a S_{ni} shifted by c_i units to the right. The first term is multiplied by unity and the subsequent terms are multiplied by the binomial of definition 2.

There exist m+1 terms in the reduced form of the vector product e.g. [18]. Transforming to the cycle index form there are Binomial[m+2,2] terms. Fortunately, once the first highest term is calculated the remaining terms are obtained by successively removing the highest m term. An example, the complete calculation of [18] in terms of the cycle index scalar product for P[7,5,n,3] is shown:

[21] $$\{1,7,21,35,35,21,7,1\}.\{792,1716,3432,6435,11440,19448,31824,50388\} + \{1,6,15,20,15,6,1\}.\{924,1716,3003,5005,8008,12376,18564\} + \{1,5,10,10,5,1\}.\{792,1287,2002,3003,4368,6188\} + \{1,4,6,4,1\}.\{495,715,1001,1365,1820\} + \{1,3,3,1\}.\{220,286,364,455\} + \{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1 + \mathrm{Binomial}[n-1,n-2] * (\{1,6,15,20,15,6,1\}.\{924,1716,3003,5005,8008,12376,18564\} + \{1,5,10,10,5,1\}.\{792,1287,2002,3003,4368,6188\} + \{1,4,6,4,1\}.\{495,715,1001,1365,1820\} + \{1,3,3,1\}.\{220,286,364,455\} + \{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n,n-2] * (\{1,5,10,10,5,1\}.\{792,1287,2002,3003,4368,6188\} + \{1,4,6,4,1\}.\{495,715,1001,1365,1820\} + \{1,3,3,1\}.\{220,286,364,455\} + \{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n+1,n-2] * (\{1,4,6,4,1\}.\{495,715,1001,1365,1820\} + \{1,3,3,1\}.\{220,286,364,455\} + \{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n+2,n-2] * (\{1,3,3,1\}.\{220,286,364,455\} + \{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n+3,n-2] * (\{1,2,1\}.\{66,78,91\} + \{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n+4,n-2] * (\{1,1\}.\{12,13\} + 1) + \mathrm{Binomial}[n+5,n-2]$$

The above analysis suggests a combinatoric role of the general Jacobi Polynomial evaluated at x = 3. We saw in Chapter 24 the connection of Delannoy numbers to the Jacobi polynomial P[m,a,0,3]. Based on the reduction of any Jacobi to a series of associated Delannoy numbers such as [21] a convoluted union of paths are likely represented. These paths suggest a connection to Brownian like motion in two dimensions.

a. Another polynomial sequence of the Jacobi polynomial P[m, a, 0, 1] can be obtained from the scalar product of the m-1 and a-1 rows of the Pascal triangle. These rows are equivalent to the diagonal of the Chebyshev S polynomial. P[6,4,0,1] = 210 = $\{1,6,15,20,15,6,1\}.\{1,4,6,4,1,0,0\}$.

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Postscript: The Jacobi polynomial can also be used for the determination of the nth Perrin number. *Mathematica* is used for the calculations. As in Chapter 15 let,

$A0 = ((N1 - 3))/2$ and $B0 = 1$

$A = \text{RecurrenceTable}[\{a[n+1] == a[n] - 3, a[0] == A0\}, a, \{n, 0, \text{Floor}[A0/3]\}]$

$B = \text{RecurrenceTable}[\{a[n+1] == a[n] + 2, a[0] == B0\}, a, \{n, 0, \text{Floor}[A0/3]\}]$

Then the hypergeometric function [Eqn15.1.20] can be transformed as

[P0] $So = \sum_{n=1}^{\text{Floor}[A0/3]+1} (1/(A[[n]] + B[[n]])) * \text{JacobiP}[A[[n]], 0, (-(A[[n]] + B[[n]]) - 1), -1]$

Or equivalently,

[P0-1] $So = \sum_{n=1}^{\text{Floor}[A0/3]+1} (-1)^n (1/(A[[n]] + B[[n]])) * \text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), 0, 1]$

where So is the Sigma orbit and the Perrin number PN is

$PN = \text{Round}[N1 * So]$

The equation is accurate for all numbers >12 (6 and 12 give one less than PN).

For prime numbers let So be expressed as the Jacobi polynomial at x = 3. Then we can write,

[P1] $So = \sum_{n=1}^{\text{Floor}[A0/3]+1} (-1)^n (1/(A[[n]] + B[[n]])) * \text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), B[[n]], 3]$

From [Eqs 4 and 19] So can be written as a sum of Jacobi polynomials;

$\text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), 0, 3]$

which can be reduced to

[P2] $\text{JacobiP}[A[[n]], ((B[[n]]) + 1) - (2 + x * 3), 0, 3] = \text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), 0, 3]$

Where x is an integer described below.

I find that the value of original Jacobi polynomial in [P1]

[P3] $\text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), B[[n]], 3]$ = Sn(i)

where Sn(i) is the ith entry of the cycle index polynomial of the symmetry group Sn.

[P4a,b] Sn $= SA[[n]]$ and I = $A[[n]] + 1 + ((B[[n]]) + 1) - (2 + x * 3)$

Example; For N = 47 the $A[[5]] = 10$ *and* $B[5] = 9$ then $\text{Floor}[A0/3] + 1 - 5 = 3 = x$

$((B[[5]]) + 1) - (2 + x * 3) = 10 - 11 = -1$

$i = A[[5]] + 1 + ((B[[5]]) + 1) - (2 + 3 * 3) = 10$

[P5] $\text{JacobiP}[10, -20, 9, 3] = S_{10}(10) = 92376$

The expression for the cycle index polynomial for any symmetry group S_m given in *Mathematica* can also be written as

[P6] $|\text{JacobiP}[m, -(m + i - 1) - 1, i - 1, x]|$ = |Binomial[m+i-1,m]|

The values of i are integers and are limiting values as x approaches 1.

All sigma orbits of prime numbers are represented by the symmetry groups represented in $A[[n]]$

For the primes 23,29,41,47,59,71,83 and 89 the entries from S_{10} are respectively,

$$S_{10}(2), S_{10}(4)\ , S_{10}(8), S_{10}(10)\ , S_{10}(12), S_{10}(14)\ , S_{10}(18), S_{10}(22), S_{10}(24)$$

Equation [P1] demonstrates another combinatoric role for the Jacobi Polynomial. It provides a direct calculation of the sigma orbit and Perrin number for any prime; and can be used to predict the value of the cycle index polynomial for symmetry groups. The limiting values of the Jacobi polynomial [P6] were found by in attempts to express the Jacobi polynomial in reduced forms. The analysis suggests that any prime is associated with the sigma orbit as a series of symmetry group entries that cannot be duplicated by composite numbers or Perrin pseudoprimes. Complex infinities are found when the Jacobi Polynomial in [P3] is used with composite numbers. These infinities are not found using [PO] or [PO-1].

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