Expressing a q Continued Fraction in Terms of Radicals

In Chapter 20, I was able to express the eta quotient as a function of the root ξ or roots of minimal irreducible cubic polynomials. Some polynomials of Class number 3 could be solved from the eta quotient as linear functions of ξ , but many of the larger discriminants required a polynomial up to fourth or fifth order in ξ. For example f(-907) = $\frac{\xi^3(\xi^2+\xi-1)}{8}$ where ξ is the root of $X^3 - 5X^2 + X - 2 = 0$. The modulus f(-d) of the eta quotient [1] is,

[1] $$f(-d) = |n(\tau 2)/n(\tau 1)| * \frac{\sqrt{2a1}}{\sqrt{2a2}}$$

with $\tau 1 = \frac{b1+\sqrt{b1^2-4a1c1}}{2a1}$ and $\tau 2 = \frac{b2+\sqrt{b2^2-4a2c2}}{2a2}$ such that -d = $b2^2 - 4a2c2$ is the discriminant of a cubic equation.

There is a deep connection of the modular function [1] with Ramanujan's continued fractions and the geometry and symmetry of the platonic solids [1,2]. One continued fraction **u(τ)** studied by Ramanujan in his famous notebooks[3] is of interest and connects q-continued fractions to the modular function [1] and roots of certain polynomials of discriminant (-d). .

[2] $$u(\tau) = \sqrt{2} * q^{1/8} * \prod_{n>1} \frac{(1-q^{2n-1})}{(1-q^{4n-2})^2} = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5 \ldots}}}}}}$$

Here q = $e^{2\pi i\tau}$.

It is shown in the literature that **u(τ)** is an eta modular function and generally a complex number.

[3] $$u(\tau) = \sqrt{2} * \frac{\eta(\tau)*(\eta(4\tau))^2}{(\eta(2\tau))^3}$$

Let the modulus of $u(\tau)$ be defined as a real number calculated from a complex number, x+iy, by multiplying by its conjugate x-iy;

[4] $$|u(\tau)| = \sqrt{2} * \frac{\eta(\tau)*(\eta(4\tau))^2}{(\eta(2\tau))^3} * \text{Conjugate}(\sqrt{2} * \frac{\eta(\tau)*(\eta(4\tau))^2}{(\eta(2\tau))^3})$$

Define U and U2 as the following conjugates,

[5] $$|U| = \frac{\eta(\tau)}{(\eta(2\tau))^1} * \text{Conjugate}(\frac{\eta(\tau)}{(\eta(2\tau))^1})$$

[6] $$|U2| = \sqrt{2} * \frac{(\eta(4\tau))^2}{(\eta(2\tau))^2} * \text{Conjugate}(\sqrt{2} * \frac{(\eta(4\tau))^2}{(\eta(2\tau))^2})$$

[7] $$|u(\tau)| = |U2| * |U|$$

The following relations are true in special cases for the conjugates u(τ), U and U2.

[8a,b] $|U|^4 * |U2| = 2$

$$U = \left(\frac{2}{U2*U}\right)^{1/3} = \left(\frac{2}{|u(\tau)|}\right)^{1/3}$$

Where U is a root of a polynomial and an algebraic integer. This equation can be generalized as,

[9] $$U(\tau) = \left(\frac{2}{|u(\tau)|}\right)^{k/3} * Q$$

where k is a positive integer and Q is a real number radical or a radical fraction. Equation [9] now relates the modulus of the q fraction in [1] to the roots of polynomials. These roots if expressed in radical form demonstrate that the q fraction for a given discriminant and values of $\tau 1$ and $\tau 2$ from equation [1] can be written as a equation of radicals.

Proposition 1 – The modulus of the q continued fraction $u(\tau)$ *, is expressible in radicals for negative discriminants 1 mod 4 and 3 mod 4 where q =* $e^{2\pi i * \tau}$ *and* $\tau = \frac{1+\sqrt{-d}}{4}$ *. Class number 3 discriminants except d=-23, -31 are not expressed by* $u(\tau)$*.*

From [9] we obtain;

[10] $$|u(\tau)| = 2\left(\frac{U(\tau)}{Q}\right)^{-3/k}$$

The following table shows values of k and Q for various discriminants. U{τ) values are obtained either as roots to irreducible polynomials from *Mathematica* or radical expressions from Weber [4].

k	Q	Discriminant	Radical Form of |u(τ)| from [10]
1	1	−1	$2^{1/4}$
1	1	−3	1
4	1	−5	$\frac{2}{(1+\sqrt{5})^{3/4}}$
1	1/2	−7	$\frac{1}{\sqrt{2}}$
3	1	−9	$\frac{2^{3/4}}{1+\sqrt{3}}$
1	1	−11	$\frac{54}{\left(2-\frac{2}{(17+3\sqrt{33})^{1/3}}+(17+3\sqrt{33})^{1/3}\right)^3}$
4	1	−13	$\frac{2}{(3+\sqrt{13})^{3/4}}$
3	1	−15	$\frac{\sqrt{2}}{1+\sqrt{5}}$
2	$1/\sqrt{2}$	−17	$\frac{8*2^{1/4}}{\left(1+\sqrt{17}+\sqrt{2(1+\sqrt{17})}\right)^{3/2}}$
1	1	−19	$\frac{18}{\left((9-\sqrt{57})^{1/3}+(9+\sqrt{57})^{1/3}\right)^3}$

12	2	−21	$\frac{2*2^{1/4}}{\sqrt{3+\sqrt{7}}(\sqrt{3}+\sqrt{7})^{3/4}}$
1	$1/\sqrt{2}$	−23	$\frac{9\sqrt{2}}{((9-\sqrt{69})^{1/3}+(9+\sqrt{69})^{1/3})^3}$
1	$\sqrt[4]{8}$	−25	$2^{1/4}(-2+\sqrt{5})$
3	1/2	−27	$\frac{1}{1+2^{1/3}+2^{2/3}}$
4	1/2	−29	$2(\frac{3}{9+\sqrt{29}+\frac{79+15\sqrt{29}}{(369+70\sqrt{29}+12\sqrt{6(27+5\sqrt{29})})^{1/3}}+2(369+70\sqrt{29}+12\sqrt{6(27+5\sqrt{29})})^{1/3}})^{3/}$
1	$1/\sqrt{2}$	−31	$\frac{27}{\sqrt{2}(1+(\frac{1}{2}(29-3\sqrt{93}))^{1/3}+(\frac{1}{2}(29+3\sqrt{93}))^{1/3})^3}$
4	1	−37	$\frac{2^{1/4}}{(6+\sqrt{37})^{3/4}}$
2	$1/\sqrt{2}$	−41	$\frac{16*2^{3/4}}{(5+\sqrt{41}+\sqrt{2(5+\sqrt{41})}+\frac{2}{\sqrt{\frac{5+\sqrt{41}}{138+18\sqrt{41}+33\sqrt{2(5+\sqrt{41})}+5\sqrt{82(5+\sqrt{41})}}}})^{3/2}}$
1	1	−43	$\frac{54}{(2+(35-3\sqrt{129})^{1/3}+(35+3\sqrt{129})^{1/3})^3}$
6	$\sqrt{2}$	−57	$\frac{2*2^{1/4}}{(1+\sqrt{3})^{3/2}\sqrt{13+3\sqrt{19}}}$
1	1	−67	$\frac{54}{(2+(53-3\sqrt{201})^{1/3}+(53+3\sqrt{201})^{1/3})^3}$
2	$1/\sqrt{2}$	−97	$\frac{8*2^{1/4}}{(9+\sqrt{97}+3\sqrt{2(9+\sqrt{97})})^{3/2}}$
6	$(\sqrt{2})^{13}$	−105	$\frac{16*2^{1/4}}{\sqrt{\sqrt{5}+\sqrt{7}}((1+\sqrt{5})(3+\sqrt{3}+\sqrt{7}+\sqrt{21}))^{3/2}}$
4	2	−133	$\frac{2(\frac{2}{5\sqrt{7}+3\sqrt{19}})^{3/4}}{(3+\sqrt{7})^{3/2}}$
1	1	−163	$\frac{54}{(6+(135-3\sqrt{489})^{1/3}+(3(45+\sqrt{489}))^{1/3})^3}$
6	$(\sqrt{2})^7$	−177	$\frac{4*2^{3/4}}{(1+\sqrt{3})^{9/2}\sqrt{23+3\sqrt{59}}}$
1	$1/\sqrt{2}$	−193	$\frac{2*2^{3/4}}{(13+\sqrt{193}+\sqrt{358+26\sqrt{193}})^{3/2}}$

As an example, consider discriminant -33 = 3 mod 4. In reference (4), Table VI the value of $U(\tau)$ is given. The Weber function (eta quotient) given as a solution is raised to the 6th power and multiplied by the square root of 2. Substituting into [10]:

$$\left|u(\frac{1+\sqrt{-33}}{4})\right| = 2\left(\frac{(3+\sqrt{11})*(1+\sqrt{3})^3}{\sqrt{2}}\right)^{-3/6} = 0.2095616440794404494992565980 19$$

A calculation from equation [2], u(τ) = $0.4489937002 + 0.0892541386i$. Multiplying by the conjugate results in the value given above. Rearranging the radicals shows that in exact form for d= -33,

$$|u(\tau)| = \frac{2*2^{1/4}}{(1+\sqrt{3})^{3/2}\sqrt{3+\sqrt{11}}}$$

In references (1,2) the connection of the q continued fractions to the tetrahedron, octahedron and icosahedron is discussed. The q continued fraction in [2] associates with the octahedron which has 12-edges, 6 - (vertices) and 8 -faces. It is interesting to note that taking the eighth power of the complex number u(τ), results in a real number which is the eighth power of |u(τ)|! This is true for all discriminants mentioned in proposition 1. It is also noted that $u(\tau)^8$-1 times its conjugate is exactly equal to 1 in all cases. The q continued fraction is said to map points τ on the upper half of the complex plane to a point u(τ) on the octahedron projected on the complex plane. Symmetry properties of the octahedron projected (mapped) onto a sphere are then preserved when projected on the complex plane. In addition, invariant properties such as the j-invariant of a polynomial with discriminant -d are also preserved on the octahedron. The octahedral equation [11] illustrates this symmetry;

[11] $$(u(\tau)^{16} + 14 * u(\tau)^8 + 1)^3 - (2^{-4}) * \mathrm{j}(\tau) * (u(\tau)^8 * (u(\tau)^8 - 1)^4) = 0$$

This equation is also true for all discriminants tested by proposition 1 and is only true for the complex values of u(τ). Rearranging [11] the j invariant is a ratio of two terms which are also equations of the octahedron defining its edges and vertices.

Many interesting properties of the morphic numbers for discriminants -23 and -31 are retained for the q continued fractions $|u\left(\frac{1+\sqrt{-23}}{4}\right)|$ and $|u\left(\frac{1+\sqrt{-31}}{4}\right)|$. The plastic number ψ is discussed in Chapter 19 on the geometry of the Perrin number. Some of these properties are shown in the expressions below.

[12] $$\frac{1}{\psi} = 2^{1/6}|u(\tau)|^{1/3}$$

[13] $$(\frac{1}{\psi})^5 = 2^{5/6}|u(\tau)|^{5/3}$$

[14] $$2^{5/6}|u(\tau)|^{5/3} + 2^{1/6}|u(\tau)|^{1/3} = 1$$

[15] $$(\frac{1}{\psi})^6 = 2|u(\tau)|^2$$

[16] $$\psi^2 = 2^{1/6}|u(\tau)|^{1/3} + 1$$

[17] $$|u(\tau)| = \frac{(\psi^2-1)^3}{\sqrt{2}}$$

I previously demonstrated how the real and complex powers of the Perrin sequence can be generated from 1, ψ, 1+1/ ψ and 2, - ψ , ψ^{-5}. The resulting sequence is

[18] $3,0,1+\frac{1}{\psi^5}+\frac{1}{\psi},3,1+\frac{1}{\psi^5}+\frac{1}{\psi},4+\frac{1}{\psi^5}+\frac{1}{\psi},4+\frac{1}{\psi^5}+\frac{1}{\psi},5+\frac{2}{\psi^5}+\frac{2}{\psi},8+\frac{2}{\psi^5}+\frac{2}{\psi},9+\frac{3}{\psi^5}+\frac{3}{\psi},13+\frac{4}{\psi^5}+\frac{4}{\psi},17+\frac{5}{\psi^5}+\frac{5}{\psi}\ldots$

There are two sequences 1, 3,1, 4, 4, 5, 8, 9, 13, 17, 22, 30…. and 1,1,2,2,3,4,5,7,9,12,16,21…. The first sequence is generated from the coefficients of $|(1-3z)/(1-z^2+z^3)|$, positive coefficients of [OEIS A117374]. The second sequence are the Padovan numbers generated from $1/(1-z^2+z^3)$ [OEIS A182097]. Combining, the Perrin sequence is generated by $|(2-3z)/(1-z^2+z^3)|$. The real root of $1 - z^2 + z^3 = 0$ is the negative inverse of the plastic number ($-\frac{1}{\psi}$). Using equation [10] I find that for discriminant -23, an algebraic number and it's inverse result in two equivalent radical forms.

[19] $$|u(\tau)| = \frac{\left(-2+2^{2/3}(25-3\sqrt{69})^{1/3}+2^{2/3}(25+3\sqrt{69})^{1/3}\right)^3}{216\sqrt{2}} = \frac{9\sqrt{2}}{((9-\sqrt{69})^{1/3}+(9+\sqrt{69})^{1/3})^3}$$

Equation [14] specified for $\left|u\left(\frac{1+\sqrt{-23}}{4}\right)\right|$ can be generalized to find other polynomials and associated q continued fractions. For example,

[20] $$2^{1/6}\left|u\left(\frac{1+\sqrt{-31}}{4}\right)\right|^{1/3} + 2^{3/6}\left|u\left(\frac{1+\sqrt{-31}}{4}\right)\right|^{3/3} = 1$$

[21] $$2^{2/3}\left|u\left(\frac{1+\sqrt{-19}}{4}\right)\right|^{1/3} - 2^{0/6}\left|u\left(\frac{1+\sqrt{-19}}{4}\right)\right|^{6/3} = 1$$

[22] $$2^{2/3}\left|u\left(\frac{1+\sqrt{-43}}{4}\right)\right|^{1/3} + 2^{0/6}\left|u\left(\frac{1+\sqrt{-43}}{4}\right)\right|^{3/3} = 1$$

Using *Mathematica*, a general *q Modulus* equation [23] can be developed for $|z|$ with assigned values of a, b, c, and d.

[23] $$2^{a/3}|z|^{b/3} + 2^{c/6}|z|^{d/3} - 1 = 0$$

Once $z = |u(\tau)|$ is found an associated root of a polynomial is calculated using equation [9] and *Mathematica* will find a root approximation to a polynomial and its discriminant. If the discriminant is 1 or 3 mod 4 then the associated $u(\tau)$ is either 1 or a radical multiple of the q continued fraction in equation [2]. Unfortunately, simple multiplication of $u(\tau)$ by a radical does not guarantee that $u(\tau)$ is a root of the octahedral equation [11]. Only, equations [14] and [20]-[22] are known to result in the radical equaling one and $u(\tau)$ satisfying equation [11].

It is desired to find conditions for which z is a real number. The associated root and polynomial will have a discriminant which is a multiple of a prime discriminant d. Once this discriminant has been determined then the appropriate $\tau_d = \frac{1+\sqrt{-d}}{4}$ is used to find the radical $R = |u(\tau_d)|/|z|$. Equation [2] is modified by multiplying by $1/\sqrt{R}$ and both the desired $u(\tau)$ and $z = |u(\tau)|$ are obtained. Since the octahedral equation is not valid when equation [2] is multiplied by another number, a new value of τ is required. The equations which are used to provide this information are the equations for the j- invariant. There are two equations, (1) $j(\tau)$ from rearranged equation [11] with the $u(\tau)$ obtained from a modified equation [2] and (2) finding the root τ from the defined equation for the j-invariant. Equation [24] provides this root τ in *Mathematica*.

[24] FindRoot[$((\lambda[2,0,E\text{^}(\pi*i*\tau)]\text{^}8+\lambda[3,0,E\text{^}(\pi*i*\tau)]\text{^}8+\lambda[4,0,E\text{^}(\pi*i*\tau)]\text{^}8)\text{^}3*1728 - j(\tau)*(54(\lambda[2,0,E\text{^}(\pi*i*\tau)]\lambda[3,0,E\text{^}(\pi*i*\tau)]\lambda[4,0,E\text{^}(\pi*i*\tau)])\text{^}8)),\{\tau,(1+\sqrt{-11})/4\}]$

In this program λ is the EllipticTheta function calculated by *Mathematica*, $E^\wedge(\pi * i * \tau)]$ is the Nome q(τ) and $(1+\sqrt{-11})/4$ is a suggested starting point for the search of the root τ .
The general method is outlined with examples below for four examples of a, b, c, and d.

(a,b,c,d) = (2,1,6,3)

Solving the q modulus equation gives z = $|u(\tau)| = \frac{1}{4}(1-(\frac{2}{3(-9+\sqrt{93})})^{1/3}+\frac{(\frac{1}{2}(-9+\sqrt{93}))^{1/3}}{3^{2/3}})$. Substituting z in [9] with k=1 and Q = 1, gives the root 2.9311424637535 …
with a root approximation for the polynomial $x^3 - 2x^2 - 8 = 0$. The discriminant d = -1984 = -2^6*31. Using $\tau_{31} = \frac{1+\sqrt{-31}}{4}$, a value of $|u\left(\frac{1+\sqrt{-31}}{4}\right)|$ is obtained from [20] and the ratio found as $|\frac{u\left(\frac{1+\sqrt{-31}}{4}\right)}{z}|/z = 2\sqrt{2}$. Multiply $1/\sqrt{2\sqrt{2}}$ with equation [2] to obtain the modified $u(\tau)$ and |z|. Use [11] to find the new value of j(τ) and plug into [24] to locate the new value of τ. In this example; τ = 0.24980 …. + 2.0538479 … i and the octagonal equation is solved with $u(\tau)$ provided by the root |z| = 0.07941804904 … to [23].

As alternative, multiply z by $2\sqrt{2}$. Then $|u(\tau)| = \frac{1-(\frac{2}{3(-9+\sqrt{93})})^{1/3}+\frac{(\frac{1}{2}(-9+\sqrt{93}))^{1/3}}{3^{2/3}}}{\sqrt{2}}$ and the octagonal equation is solved with $\tau = \frac{1+\sqrt{-31}}{4}$.

(a,b,c,d) = (2,1,10,5)

Solving the q modulus equation gives z = $|u(\tau)| = \frac{1}{12}\left(2-5\left(\frac{2}{-11+3\sqrt{69}}\right)^{1/3}+\left(\frac{1}{2}\left(-11+3\sqrt{69}\right)\right)^{1/3}\right)$. Substituting z in [9] with k=1 and Q = 1, gives the root 2.6494359142 …with a root approximation for the polynomial $x^3 - 4x^2 - 8 = 0$. The discriminant d = -1472 = -2^6*23. Using $\tau_{23} = \frac{1+\sqrt{-23}}{4}$, a value of $|u\left(\frac{1+\sqrt{-23}}{4}\right)|$ is obtained from [14] and the ratio $|\frac{u\left(\frac{1+\sqrt{-23}}{4}\right)}{z}|/z = 2\sqrt{2}$ is found. Multiply $1/\sqrt{2\sqrt{2}}$ with equation [2] to obtain the modified $u(\tau)$ and |z|. Use [11] to find the new value of j(τ) and plug into [24] to locate the new value of τ. In this example; τ = 0.249329 …. + 1.86086518 … i and the octagonal equation is solved with $u(\tau)$ and provided by the root |z| = 0.107539927 … to [23]. As alternative, multiply z by $2\sqrt{2}$. Then $|u(\tau)| = \frac{2-5(\frac{2}{-11+3\sqrt{69}})^{1/3}+(\frac{1}{2}(-11+3\sqrt{69}))^{1/3}}{3\sqrt{2}}$ and the octagonal equation is solved with $\tau = \frac{1+\sqrt{-23}}{4}$.

(a,b,c,d) = (6,3,18,9)

Solving the q modulus equation gives z = $|u(\tau)| = \frac{1}{4}(-(\frac{2}{3(9+\sqrt{93})})^{1/3}+\frac{(\frac{1}{2}(9+\sqrt{93}))^{1/3}}{3^{2/3}})$. Substituting z in [9] with k=1 and Q = 1, gives the root 2.271776692 …with a root approximation for the polynomial $x^9 - 8x^6 - 512 = 0$. The discriminant d = $-2^{72}*3^9*31^3$. Using $\tau_{31} = \frac{1+\sqrt{-31}}{4}$, a value of $|u\left(\frac{1+\sqrt{-31}}{4}\right)|$ is obtained from [20] and the ratio found as $|\frac{u\left(\frac{1+\sqrt{-31}}{4}\right)}{z}|/z = \frac{2}{\sqrt{\frac{3}{4-22(\frac{2}{-47+9\sqrt{93}})^{1/3}+2^{2/3}(-47+9\sqrt{93})^{1/3}}}}$. Multiply 1/square root of this ratio with equation [2] to obtain the modified $u(\tau)$ and |z|. Use [11] to find the new value of j(τ) and plug into [24] to locate the new value of τ. In this example; τ = 0.249329 …. +

$1.86086518\ldots i$ and the octagonal equation is solved with $u(\tau)$ and provided by the root |z| = $0.17058195\ldots$ to [23]. As alternative, multiply z by $\frac{2}{\sqrt{\frac{3}{4-22(\frac{2}{-47+9\sqrt{93}})^{1/3}+2^{2/3}(-47+9\sqrt{93})^{1/3}}}}$. Then $|u(\tau)| = \frac{(-2(\frac{6}{9+\sqrt{93}})^{1/3}+2^{2/3}(9+\sqrt{93})^{1/3})\sqrt{4-22(\frac{2}{-47+9\sqrt{93}})^{1/3}+2^{2/3}(-47+9\sqrt{93})^{1/3}}}{12*3^{1/6}}$ and the octagonal equation is solved with $\tau = \frac{1+\sqrt{-31}}{4}$.

(a,b,c,d) = (3,3,6,6)
Solving the q modulus equation gives z = $|u(\tau)| = \frac{1}{4}(-1+\sqrt{5})$. Substituting z in [9] with k=1 and Q = 1, gives the root $1.86358501876\ldots$with a root approximation for the polynomial $x^6 - 4x^3 - 16 = 0$. The discriminant d = $-2^{20}*3^6*5^3$. Using $\tau_5 = \frac{1+\sqrt{-5}}{4}$, a value of $|u\left(\frac{1+\sqrt{-5}}{4}\right)|$ is obtained and the ratio found as $|\frac{u\left(\frac{1+\sqrt{-5}}{4}\right)}{z}|/z = 2(1+\sqrt{5})^{1/4}$. Multiply 1/square root of this ratio with equation [2] to obtain the modified $u(\tau)$ and |z|. Use [11] to find the new value of j(τ) and plug into [24] to locate the new value of τ. In this example; τ = $0.212776948\ldots. + 1.18872246\ldots i$ and the octagonal equation is solved with $u(\tau)$ and provided by the root |z| = $0.30901699437\ldots$ to [23]. As alternative, multiply z by $2(1+\sqrt{5})^{1/4}$. Then $|u(\tau)| = \frac{1}{2}(-1+\sqrt{5})(1+\sqrt{5})^{1/4}$ and the octagonal equation is solved with τ = $\frac{1+\sqrt{-5}}{4}$.

The q modulus equation can also be used to find the q continued fraction for the other class number 3 discriminants that could not be readily calculated using Proposition 1. Extending the modulus equation to these discriminants requires adding extra terms to [23]. The general equation is shown in [25];

[25] $$2^{a/3}|z|^{b/3} + N * 2^{c/6}|z|^{d/3} + M * 2^{e/6}|z|^{f/3} - I = 0$$

Where |z| is the q modulus, a, b, c, d, e and f and N, M and I are positive or negative integers.

*Proposition 2 – The modulus |z| of the q continued fraction $u(\tau)$, is expressible in radicals for **all** negative discriminants 1 mod 4 and 3 mod 4 where q = $e^{2\pi i*\tau}$. The modulus is calculated from the q modulus equation [25] and τ is determined as described in the procedure above. The root of the associated polynomial is obtained from [9] with k= Q = 1.*

Let (a, b, c, d, e, f,/ N, M, I) represent the constants in [25]. There can be multiple solutions to this equation and each solution is considered separately. An example is given below.

(a, b, c, d, e, f,/ N, M, I) =(-3, 3, 2, 1, 4, -1,/ -1, 1, 5) for d = -83
Solving the q modulus equation[25] gives two roots, z = $|u(\tau)| = \frac{2}{3}(7 - \frac{10}{(-46+3\sqrt{249})^{1/3}} + 2(-46+3\sqrt{249})^{1/3})$ and 16. Substituting the first radical solution z in [9] with k=1 and Q = 1, gives the root $2.8311772\ldots$with a root approximation for the polynomial $x^3 - 2x^2 - 2x - 1 = 0$. The discriminant d = -83 is a class number 3 discriminant. Using $\tau_{83} = \frac{1+\sqrt{-83}}{4}$, a value of $|u\left(\frac{1+\sqrt{-83}}{4}\right)|$ is obtained and the

ratio found as $|\frac{u\left(\frac{1+\sqrt{-83}}{4}\right)}{z}|$/z is calculated. Multiply 1/square root of this ratio with equation [2] to obtain the modified $u(\tau)$ and |z|. Use [11] to find the new value of j(τ) and plug into [24] to locate the new value of τ. In this example; τ = $0.250004024 \ldots. + 1.98757645 \ldots i$ and the octagonal equation is solved with $u(\tau)$ and provided by the root |z| = 0.08813102796 ... to [25]. Following the same procedure for the second solution of 16; τ = $-0.250000438 \ldots. + 2.206356 \ldots i$ and the octagonal equation is solved with $u(\tau)$ and provided by the root |z| = 16 to [25]. The procedure shows however that 1/ $|u(-0.250000438 \ldots. +2.206356 \ldots i)|$ = 16. The reason for the inverse is currently unknown since the root |z| = 16 and not 1/16 satisfies [25]. The reflection of τ across the imaginary axis appears to invert the root.

As another example the representation for d= -907 which was mentioned at the beginning of this chapter is
(3, 3, -1, 1, 1, -1,/ 9, -2, -3). The one solution obtained is shown in [26] with its calculated τ value.

[26] $|u(0.2500000069964675 + 3.027035522597698i)| = \frac{1}{12}(-17 - \frac{2495}{(66943+2712\sqrt{2721})^{1/3}} + (66943 + 2712\sqrt{2721})^{1/3})$

Substituting this result into [9] with k=Q=1, results in the real root of x^3-5x^2+x-2 = 0 as expected. Note that for class 3 discriminants the ratios found above cannot be converted to simple radical form using $\frac{1+\sqrt{-d}}{4}$ (see next section below). A cusp appears about the discriminant $\frac{1+\sqrt{-d}}{4}$. It is also noted that $u(\tau)^8$-1 times its conjugate is not exactly equal to 1 unless the discriminant is exactly $\frac{1+\sqrt{-d}}{4}$.

As found in these 5 examples the calculated value for τ remains in the upper complex plane and maps $u(\tau)$ to the octahedron projected on the complex plane. It would be interesting to find if a rotational matrix exists to transform the complex quadratic fields $\frac{1+\sqrt{-d}}{4}$ to the value found for τ. *Mathematica* provides the tools to express the q modulus equation from the octic q continued fraction in radical form for a variety of quadratic fields.

Irreducible polynomials and Adjunction

In the analysis above, many of the roots to $|u(\tau)|$ and its associated root $U(\tau)$ from equation [9] above are roots of polynomials with order n < 5. In many cases as shown in the Table above, radical solutions can be found. Except for d= -23 and d= -31 the class number 3 discriminants resulted in values of $|u\left(\frac{1+\sqrt{-d}}{4}\right)|$ which could not be roots of a polynomial of order less than 5 for any value of k and Q. However, we know that all the class number 3 discriminants are cubic polynomials. For this reason, a numerical method was used to find the nome q for which the value of $|u\left(\frac{1+\sqrt{-d}}{4}\right)|$ could be found in radical notation for a known irreducible polynomial. The result as demonstrated in equation [26] for d = -907 showed that the quadratic needed to solve $|u\left(\frac{1+\sqrt{-907}}{4}\right)|$ was not $\frac{1+\sqrt{-907}}{4}$ but the complex number $0.2500000069964675.. + 3.027035522597698..i$. This resulted in the root of the correct irreducible polynomial but does not provide the true value for $u\left(\frac{1+\sqrt{-907}}{4}\right)$ and its modulus. (The value of the LHS of equation [26] is $0.0172195640049\ldots$ but$\left| u\left(\frac{1+\sqrt{-907}}{4}\right)\right|$ = $0.000014613695463..$ a 1200- fold difference in value!

As indicated above we seek a radical form of $|\ u(\tau)\ |$ for which the complex number $u(\tau)^8$-1 times its conjugate is exactly unity. In many cases this is difficult if the value of U is a root of a polynomial of order >4 and the polynomial is irreducible. This is expressed by the Abel-Ruffini theorem that the *general polynomial of degree n greater or equal to 5 is not solvable by radicals.* But, discriminants of class number 3 and many other discriminants result in orders > 5 which can be solved by radicals! This was known in Weber's time and many of his entries in Table VI of his treatise demonstrate this.

The Abel-Ruffini theorem applies to general polynomials with rational constant coefficients. In most cases these coefficients are integers and the polynomial cannot be reduced to a product of polynomials of lesser order. In this situation we are dealing with an integral domain of characteristic 0 and not of a prime p so mx=0 only if m = 0 for all integers x. If the field of coefficients is extended by a radical field, then any radical subfield containing the integer coefficients (squares) can be divided by the radical field. We indicate this by allowing radical coefficients to a general (monic) polynomial $x^n + r_1x^{(n-1)} + \ldots.. r_n = 0$. The x values are said to be algebraic integers over this field containing r_n. The splitting of the polynomial into multiple rational polynomials by the adjunction of a radical $\sqrt{R}$ may provide a solution to polynomials of order greater than 5.

Example d = -65.

A calculation of $\left| u\left(\frac{1+\sqrt{-65}}{4}\right)\right|$ from equation [2] results in an algebraic integer which is a root approximation to an order 35 polynomial according to Mathematica. Using equation [9] above with k = 2 and Q = $1/\sqrt{2}$ the resulting value U is a root of the polynomial $1 - 8x + 12x^2 + 8x^3 - 27x^4 + 8x^5 + 12x^6 - 8x^7 + x^8 = 0$. This polynomial is irreducible in the integers. We seek a splitting field that can split the equation into a product of two fourth order equation. The numerical solutions provided by Mathematica are four real solutions z_1 =U, z_2, z_3 and z_4 and two complex solutions with their conjugates. It is found by trial and error that

[27] $$(z- z_1)^* (z- z_2)^* (z- z_3)^* (z- z_4) = z^4 - 4z^3 - r_1z^2 - 4z - 1 = z^4 - 4z^3 - (2+\sqrt{65})\, z^2 - 4z - 1$$

This fourth order equation can be solved to provide a radical solution to the root U. Substituting this root into [10] gives the desired result:

[28] $$\left| u\left(\frac{1+\sqrt{-65}}{4}\right)\right| = \frac{2*2^{3/4}}{\left(2+\sqrt{8+\sqrt{65}}+\frac{\sqrt{(2072+257\sqrt{65})(4+\sqrt{8+\sqrt{65}})}}{(8+\sqrt{65})^{5/4}}\right)^{3/2}}$$

The method can be used to add missing values to Weber's Table in reference 4. For discriminants 1 and 3 mod 4 less than 100, all but four discriminants can be solved. Alternate methods or further research on d = 47, 71,79 and 89 is required. Although 47 and 71 are listed in Weber's table, the 5th order and 7th order polynomials, respectively could not be reduced. The two discriminants 79 and 89 are polynomials of order 5 and 12 respectively.

For class number 3 discriminants the resulting U are always roots to a ninth order polynomial. Unfortunately, the radical solutions are quite unwieldy but accurate. I close this chalkboard with an example.

Example d = -59.

A calculation of | $u\left(\frac{1+\sqrt{-59}}{4}\right)$ | from equation [2] results in an algebraic integer which is a root approximation to an order 9 polynomial according to Mathematica. Using equation [9] above with k = 1 and Q = 1 (these values are used for all class number 3 discriminants) the resulting value U is a root of the polynomial

$-8 + 16x - 8x^2 + 4x^3 - 8x^4 + 4x^5 - 2x^6 + 4x^7 - 4x^8 + x^9 = 0$. This polynomial is irreducible in the integers. We seek a splitting field that can split the equation into a product with a third order equation. The numerical solutions provided by Mathematica are one real solutions z_1 =U, and four complex solutions z_2, z_4, z_6 and z_8 with their conjugates z_3, z_5, z_7 and z_9. It is found by trial and error that

[29] $(z - z_1)*(z - z_6)*(z - z_7) = z^3 - r_1 z^2 + r_2 z - 2$

This third order equation can be solved to provide a radical solution to the root U once a radical form for r_1 and r_2 are found. These numbers always are solutions to another third order equation. I find r_1 = root$[-16 - 4x - 4x^2 + x^3]$ and r_2 = root$[-8 - 4x^2 + x^3]$ Substituting the root U in radical form into [10] gives the desired result in nested radical form:

[30] | $u\left(\frac{1+\sqrt{-59}}{4}\right)$ | =

$$2/\Big(\frac{1}{9}(4+2^{2/3}R^{1/3}+2^{2/3}S^{1/3})+$$

$$1/\Big(3\Big(2/\Big(\frac{290}{3}-2*2^{2/3}W^{1/3}-\frac{4}{9}*2^{2/3}R^{1/3}+\frac{16}{9}*2^{1/3}R^{2/3}-8T^{1/3}-2*2^{2/3}Y^{1/3}-\frac{4}{9}*2^{2/3}S^{1/3}+\frac{8}{9}R^{2/3}S^{1/3}+\frac{16}{9}*2^{1/3}S^{2/3}+\frac{8}{9}R^{1/3}S^{2/3}-8V^{1/3}-2*2^{2/3}Z^{1/3}-2*2^{2/3}X^{1/3}+\surd\Big(\frac{4}{81}$$

$$(-435+9*2^{2/3}W^{1/3}+36T^{1/3}+9*2^{2/3}Y^{1/3}+2*2^{2/3}S^{1/3}-8*2^{1/3}S^{2/3}+36V^{1/3}+9*2^{2/3}Z^{1/3}+9*2^{2/3}X^{1/3}-4R^{2/3}(2*2^{1/3}+S^{1/3})+2R^{1/3}(2^{2/3}-2S^{2/3}))^2+$$

$$4\Big(-\frac{1}{9}(4+2^{2/3}R^{1/3}+2^{2/3}S^{1/3})^2+2(2+T^{1/3}+V^{1/3})\Big)^3\Big)\Big)\Big)^{1/3}\Big)+\frac{1}{3}\Big(\frac{1}{9}(4+2^{2/3}R^{1/3}+2^{2/3}S^{1/3})^2-2(2+T^{1/3}+V^{1/3})\Big)$$

$$\Big(2/\Big(\frac{290}{3}-2*2^{2/3}W^{1/3}-\frac{4}{9}*2^{2/3}R^{1/3}+\frac{16}{9}*2^{1/3}R^{2/3}-8T^{1/3}-2*2^{2/3}Y^{1/3}-\frac{4}{9}*2^{2/3}S^{1/3}+\frac{8}{9}R^{2/3}S^{1/3}+\frac{16}{9}*2^{1/3}S^{2/3}+\frac{8}{9}R^{1/3}S^{2/3}-8V^{1/3}-2*2^{2/3}Z^{1/3}-2*2^{2/3}X^{1/3}+$$

$$\surd\Big(\frac{4}{81}(-435+9*2^{2/3}W^{1/3}+36T^{1/3}+9*2^{2/3}Y^{1/3}+2*2^{2/3}S^{1/3}-8*2^{1/3}S^{2/3}+36V^{1/3}+9*2^{2/3}Z^{1/3}+9*2^{2/3}X^{1/3}-4R^{2/3}(2*2^{1/3}+S^{1/3})+2R^{1/3}(2^{2/3}-2S^{2/3}))^2+4$$

$$\Big(-\frac{1}{9}(4+2^{2/3}R^{1/3}+2^{2/3}S^{1/3})^2+2(2+T^{1/3}+V^{1/3})\Big)^3\Big)\Big)\Big)^{1/3}\Big)^3$$

with

$R = (43 - 3\sqrt{177})$ $S = (43 + 3\sqrt{177})$ $T = (44 - 3\sqrt{177})$ $V = (44 + 3\sqrt{177})$

$W = (3485 - 261\sqrt{177})$ $X = (3485 + 261\sqrt{177})$ $Y = (299 - 3\sqrt{177})$ $Z = (299 + 3\sqrt{177})$

All class number 3 discriminants are solvable by the equations used in this example.

1. W. Duke, Continued Fractions and Modular Functions, **Bulletin of the American Mathematical Society**, **42** (2005), 137-162.
2. T. Piezas, Ramanujan's Continued Fractions and the Platonic Solids**,** https://sites.google.com/site/tpiezas/0015.
3. S. Ramanujan, Notebooks (2 volumes). Tata Institute of Fundamental Research, Bombay, 1957.
4. H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.

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