Calculating the Class Invariant using Ramanujan's Octic q Continued Fraction

Several of Weber's entries in his directory of class invariants (H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.) provide a direct solution for the eta quotient in radical form,

[1] $$f(-d) = |n(\tau 2)/n(\tau 1)| * \frac{\sqrt{2a1}}{\sqrt{2a2}}$$

In many cases the radical form of the root shown in Weber does not match the radical root form obtained by the method described in Chapter 28 for values of τ in the upper half complex plane. This Appendix shows in detail the steps required to prove equivalent forms of the radical of $f(-d)$ with $\tau = \frac{1+\sqrt{-d}}{4}$. This problem demonstrates an equivalence of radicals and nested radicals which in many cases are not obvious.

The example used in this appendix is taken from Weber's entry for d = 105. He shows that

[2] $$\sqrt{2}^{13} f(-d)^6 = (1+\sqrt{5})^3 * (1+\sqrt{3})^3 * (\sqrt{3}+\sqrt{7})^3 * (\sqrt{5}+\sqrt{7})$$

The RHS of this equation is a large number, 283054.0041056…. The number 105 factors into the product 3*5*7 and these numbers are all introduced into the solution. This is true for many of the larger discriminants but there can be many permutations and requires trial and error to obtain the complete set of factors. A rearrangement of this equation suggests that the solution may also be obtained by taking a cube root;

[3] $$\frac{\sqrt{2}^1}{(\sqrt{5}+\sqrt{7})} f(-d)^6 = (1+\sqrt{5})^3 * \left(1+\sqrt{3}\right)^3 * \left(\sqrt{3}+\sqrt{7}\right)^3 / \sqrt{2}^{12}$$

[4] $$\frac{\sqrt{2}^1}{(\sqrt{5}+\sqrt{7})}^{1/3} f(-d)^2 = (1+\sqrt{5})^1 * \left(1+\sqrt{3}\right)^1 * \left(\sqrt{3}+\sqrt{7}\right)^1 / \sqrt{2}^4$$

The form of this equation is perfect for using Ramanujan's octic q continued fraction!

[5] $$u(\tau) = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5\ldots\ldots}}}}}}$$

where q $= e^{2\pi i\tau}$ and $\tau = \frac{1+\sqrt{-105}}{4}$.

Taking the conjugate to get $|u(\tau)|$ substitute into the equation for U derived in Chapter 28,

[6] $$U(\tau) = \left(\frac{2}{|u(\tau)|}\right)^{k/3} * Q$$

where k = 2 and Q = $\frac{\sqrt{2}^1}{(\sqrt{5}+\sqrt{7})}^{1/3}$ where $|u(\tau)| = 0.035763731866302149554 94\ldots\ldots$ This algebraic integer is a root of an order 23 polynomial according to *Mathematica* so it is better to use the algebraic integer from [6] since it is a polynomial of 8th order. I find $U(\tau) = 9.676148910338828\ldots\ldots$ one root to [7];

[7] $$16 - 48z - 168z^2 + 72z^3 + 248z^4 + 36z^5 - 42z^6 - 6z^7 + z^8 = 0$$

As shown at the end of Chapter 28, it is possible to solve this equation using the adjunction of a radical to a lower order polynomial (< 5) for the integer coefficients.

Mathematica provides the numerical solution resulting in 8 real numbers, four positive solutions z_1 =U, z_2, z_3 and z_4 and 4 negative. Using the four positives

[8] $(z- z_1)*(z- z_2)*(z- z_3))*(z- z_4) = z^4 - r_1z^3 + r_2z^2 - r_3z + 4 = z^4 - (3+\sqrt{105})z^3 - (27+\sqrt{105})z^2 - (6+2\sqrt{105})z + 4 = 0$

(It is always nice when the radical form is in $\sqrt{d}$)

I find that the root in radical form is

[9] $$U = \frac{1}{4}(3+\sqrt{105}) + \frac{1}{2}\sqrt{\frac{1}{2}(11+\sqrt{105})} + \frac{1}{2}\sqrt{-\frac{11}{2} - \frac{\sqrt{105}}{2} + \frac{3}{4}(-3-\sqrt{105})^2 - 2(27+\sqrt{105}) + \frac{-8(-6-2\sqrt{105})+(-3-\sqrt{105})(-(-3-\sqrt{105})^2+4(27+\sqrt{105}))}{2\sqrt{2(11+\sqrt{105})}}}$$

This looks nothing like the RHS of [4] at this stage!

Let's make it more complicated by taking the cube of U and multiplying by 1/Q. Using the **Simplify** function on *Mathematica*,

[10] $$U^3 * \frac{1}{Q} = \frac{(\sqrt{5}+\sqrt{7})(3+\sqrt{105}+\sqrt{2(11+\sqrt{105})}+\frac{2}{\sqrt{\frac{11+\sqrt{105}}{496+48\sqrt{105}+69\sqrt{2(11+\sqrt{105})}+7\sqrt{210(11+\sqrt{105})}}}})^3}{64\sqrt{2}}$$

The numerical value of [10] is exactly the RHS of [2] divided by $\sqrt{2}^{13}$ which is $f(-105)^6$.

Several aspects of this equation are noted 1.) $\sqrt{2}^{13} = 64\sqrt{2}$ in the denominator, 2.) $(\sqrt{5}+\sqrt{7})$ remains in the numerator to the 1st power and 3.) the remaining terms are cubed. Comparison with [2] we need to show that,

[11] $$(3+\sqrt{105}+\sqrt{2(11+\sqrt{105})}+\frac{2}{\sqrt{\frac{11+\sqrt{105}}{496+48\sqrt{105}+69\sqrt{2(11+\sqrt{105})}+7\sqrt{210(11+\sqrt{105})}}}})^1 = (1+\sqrt{5})^1 * (1+\sqrt{3})^1 * (\sqrt{3}+\sqrt{7})^1$$

Expanding the RHS $=3+\sqrt{3}+3\sqrt{5}+\sqrt{7}+\sqrt{15}+\sqrt{21}+\sqrt{35}+\sqrt{105}$ we see that first term $3+\sqrt{105}$ is represented. If we then calculate the value of $\sqrt{2(11+\sqrt{105})}$ it is exactly $\sqrt{7}+\sqrt{15}$. This results in the following equality based on elimination.

[12] $$\frac{2}{\sqrt{\frac{11+\sqrt{105}}{496+48\sqrt{105}+69\sqrt{2(11+\sqrt{105})}+7\sqrt{210(11+\sqrt{105})}}}} = 3\sqrt{5}+\sqrt{3}+\sqrt{21}+\sqrt{35}$$

One trick to obtain nested radicals is expanding the square and then taking the square root. If this is performed with the RHS we obtain

[13] $$3\sqrt{5}+\sqrt{3}+\sqrt{21}+\sqrt{35} = 2\sqrt{26+9\sqrt{7}+5\sqrt{15}+2\sqrt{105}}$$

Remove the "2" from [12] and [13] and invert the LHS of [12].

[14] $$\sqrt{\frac{496+48\sqrt{105}+69\sqrt{2(11+\sqrt{105})}+7\sqrt{210(11+\sqrt{105})}}{11+\sqrt{105}}} = \sqrt{26+9\sqrt{7}+5\sqrt{15}+2\sqrt{105}}$$

If the radicand is expanded

[15] $$\sqrt{(69\sqrt{\frac{2}{11+\sqrt{105}}}+7\sqrt{\frac{210}{11+\sqrt{105}}}+\frac{496}{11+\sqrt{105}}+\frac{48\sqrt{105}}{11+\sqrt{105}})} = \sqrt{26+9\sqrt{7}+5\sqrt{15}+2\sqrt{105}}$$

Evaluating the terms under the radical I find that

[16] $$\frac{496}{11+\sqrt{105}}+\frac{48\sqrt{105}}{11+\sqrt{105}} = (-315+33\sqrt{105})+(341-31\sqrt{105}) = 26+2\sqrt{105}$$

The remaining terms on the LHS can be squared, expanded and the square root then gives,

[17] $$(69\sqrt{\frac{2}{11+\sqrt{105}}}+7\sqrt{\frac{210}{11+\sqrt{105}}})^2 = \frac{19812}{11+\sqrt{105}}+\frac{1932\sqrt{105}}{11+\sqrt{105}} \text{ and } \sqrt{(\frac{19812}{11+\sqrt{105}}+\frac{1932\sqrt{105}}{11+\sqrt{105}})} = \sqrt{942+90\sqrt{105}}$$

Doing the same with the remaining terms on the RHS of [15]

[18] $$(5\sqrt{15}+9\sqrt{7})^2 = 942+90\sqrt{105}$$

Taking the square root of the RHS of [18] and comparing with the RHS of [17] and [16] proves that [15] is satisfied and the root solution [10] is equivalent to Weber's solution $f(-d)^6$.

Using [9] two equivalent forms of $|u(\tau)|$ are found.

[19a] $$|u(\tau)| = \frac{16*2^{1/4}}{\sqrt{\sqrt{5}+\sqrt{7}}((1+\sqrt{5})(3+\sqrt{3}+\sqrt{7}+\sqrt{21}))^{3/2}}$$ from Weber

[19b] $$|u(\tau)| = \frac{16*2^{1/4}}{\sqrt{\sqrt{5}+\sqrt{7}}(3+\sqrt{105}+\sqrt{2(11+\sqrt{105})}+\frac{2}{\sqrt{\frac{11+\sqrt{105}}{496+48\sqrt{105}+69\sqrt{2(11+\sqrt{105})}+7\sqrt{210(11+\sqrt{105})}}}})^{3/2}}$$ from root of $f(-d)$

The approach in this appendix can be used to find other class invariants that are not listed by Weber or other sources on the subject. The use of equation [6] with the value of the modulus of the q continued fraction provides some extra latitude for finding a suitable irreducible polynomial with various values of k and Q. For the discriminants of class number 3, these invariants all originated from the root of a 9^{th} order monic polynomial having a constant coefficient of -8 and other coefficients as multiples of 2, 4 and 8. The two unsolved class number 5 discriminants, -47 and -71 were represented by 5^{th} order polynomials which could not be solved by adjunction. However, Q values were not explored beyond powers of 2 and $\sqrt{2}$. Other higher discriminants equal to 1 or 3 mod 4 will be tested to refine some of the analytical methods used in this paper.

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