Expressing the Octic q Continued Fraction in Radical Form using a modified Ramanujan's Class Invariant Method

Ramanujan provides in his notebooks (S. Ramanujan, *Notebooks (2 Volumes),* Tata Institute of Fundamental Research, Bombay 1957. MR 20:6340 and S. Ramanujan, *Modular Equations and Approximations to Pi*, Quarterly J. Math. (Oxford) **45**, (1914), 350-372) many new class invariants not found in Weber's directory of class invariants (H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.). The methods Ramanujan used to obtain these invariants in radical form has eluded mathematicians however many theorems using modular forms have been provided to obtain most of his published class invariants.

In his papers Ramanujan defines a class invariant G_n that provides a direct solution in the form of the eta quotient,

[1] $$G_n = f(-n) = 2^{-1/4}\,|n(\tau 2)/n(\tau 1)|$$

Fortunately, the quadratic forms of the radical of $f(-n)$ with $\tau 2 = \frac{1+\sqrt{-n}}{4}$ $or\ \tau 2 = 1+\sqrt{-n}$ and $\tau 1 = \frac{1+\sqrt{-n}}{2}$ $or\ \tau 1 = \sqrt{-n}$ give equivalent results for G_n. This equivalence provides a direct comparison of methods used in Chapter 28 with published theorems of modular forms. In this appendix I use the analysis of Berndt, et. al. (B. Berndt, H.H. Chan, and L. Zhang, *Ramanujan's Class Invariants Kronecker's Limit Formula, and Modular Equations*, Trans. American Mathematical Society, **349**(6), June 1997, 2125-2173.)

Two theorems are provided by Berndt which are used to solve several class invariants provided by Ramanujan. First, I quote these two theorems. I will then show how these theorems can be put in the form of Ramanujan's octic q continued fraction. Following Berndt, I show new radical forms which do not require solving the P-Q modular forms he proves from Ramanujan's notebooks. The P-Q modular equations used by Berndt are of degrees 3, 5 and 7. (B. Berndt, *Ramanujan's Notebooks Part III,* Springer-Verlag, New York, 1991, MR**92j**:01069).

Theorem 3.1 *(Berndt, pg 2131) Let m = 1 (mod 4) where m is a positive square free integer with prime divisor p. Let K = Q($\sqrt{-m}$) be an imaginary quadratic field such that each genus contains exactly two ideal classes and such that the principle genus G_0 contains the classes [1,Ω] and [2p, p+Ω]. Let G_1 be a nonprincipled genus containing two classes [2,1+Ω] and [p,Ω]. Then*

[2] $$\left(\frac{G_m}{G_{\frac{m}{p^2}}}\right)^{h/2} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2}$$

where h, h_1, and h_2 are the class numbers of K, Q($\sqrt{d_1}$) and Q($\sqrt{d_2}$), respectively, w and w_2 are the number of roots of unity in K and Q($\sqrt{d_2}$), respectively, ε_1 is the fundamental unit in Q($\sqrt{d_1}$) and the product is over all characters χ with $\chi(G_1) = -1$, associated with the decomposition d =d1d2, and therefore d_1,d_2,h_2,w_2 and ε_1 are dependent on χ.

Theorem 3.2 *(Berndt, pg 2132) Let m = 1 (mod 4) where m is a positive square free integer with prime divisor p. Let K = Q($\sqrt{-m}$) be an imaginary quadratic field such that each genus contains exactly two* *ideal classes and such that the principle genus G_0 contains the classes [1,Ω] and [p,Ω]. Let G_1 be a nonprincipled genus containing two classes [2,1+Ω] and [2p, p+Ω]. Then*

[3] $$(G_m G_{\frac{m}{p^2}})^{h/2} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2}$$

where h, h_1, and h_2 are the class numbers of K, Q($\sqrt{d_1}$) and Q($\sqrt{d_2}$), respectively, w and w_2 are the number of roots of unity in K and Q($\sqrt{d_2}$), respectively, ε_1 is the fundamental unit in Q($\sqrt{d_1}$) and the product is over all characters χ with $\chi(G_1) = -1$, associated with the decomposition d =d1d2, and therefore d_1,d_2,h_2,w_2 and ε_1 are dependent on χ.

In these theorems the G_m and G $_{m/p2}$ are the class invariants from equation [1] above where I define m = $\sqrt{-m}$ and m/p² = $\sqrt{-\frac{m}{p^2}}$. The class [a, b+Ω] is defined for the quadratic field z = (b + Ω)/a where Ω is $\sqrt{d_1}$ and d_1 is factor of -4m.

The form of equations [2] and [3] is perfect for using Ramanujan's octic q continued fraction.

[4] $$u(\tau) = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5\ldots\ldots}}}}}}$$

where q =$e^{2\pi i\tau}$ and $\tau = \frac{1+\sqrt{-m}}{4}$ or $\tau = \frac{1+\sqrt{-\frac{m}{p^2}}}{4}$.

Taking the conjugate to get $|u(\tau)|$ substitute into the q cubic solution equation (qkQ) for U derived in Chapter 28,

[5] $$U(\tau) = \left(\frac{2}{|u(\tau)|}\right)^{k/3} /Q$$

where k = 2 and Q = $\sqrt{2}$. Then the solution of the eta quotient from G_m is

[6] $$G^2_m = \left(\frac{2}{|u\left(\tau=\frac{1+\sqrt{-m}}{4}\right)|}\right)^{k/3} /\sqrt{2}$$

[7] $$G^2_{m/p^2} = \left(\frac{2}{|u\left(\tau=\frac{1+\sqrt{-\frac{m}{p^2}}}{4}\right)|}\right)^{k/3} /\sqrt{2}$$

The LHS of the equations found in the two theorems can be found by a ratio or product of the modulus of the octic q continued fraction!

[8] $$\frac{G_m}{G_{\frac{m}{p^2}}} = (|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4})|/|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}$$

[9] $$G_m G_{\frac{m}{p^2}} = 2^{1/6}(|u(\frac{1+\sqrt{-m}}{4})| * (|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4}))^{-1/3}$$

To begin calculation of G_m in radicals the fundamental unit ε_1 is found by examining the class structure of -4m. For this analysis we are only interested in values of -4m for which p is a prime factor and p = 3, 5, or 7. I will provide two examples from Berndt using p = 5 and 7 where one fundamental unit can be derived from d_1. The first example m = -205. -4m = - 4*5*41 and d_1= 5 = -4m/(4*41) and Theorem 3.2 is used. For the second example m = -301. -4m = -4*7*43 and d_1 = 28 = -4m/43.

In the first case we use the class $[2, 1+\sqrt{-5}]$ from the non-principle genus to find a radical from the Weber equation [1] giving the fundamental unit $\frac{1}{2}(1+\sqrt{5})$.

In the second case we use the class $[7, \sqrt{-28}]$ from the non-principle genus to find a radical from the Weber equation [1] giving the fundamental unit $8+3\sqrt{7}$.

The class numbers of K ,h, h_1, and h_2 are given in Berndt's paper as well as w and w_2. All discriminants are of class number h = 8 and w = 2.

As a conclusion to this chapter I propose a theorem stating conditions in which G_n is solvable.

Example d = -205

Using $\tau = \frac{1+\sqrt{-205}}{4}$ and $\tau p = \frac{1+\sqrt{-\frac{205}{5^2}}}{4}$ the associated octic q continued fractions are calculated to obtain $|\, u\left(\frac{1+\sqrt{-205}}{4}\right) = u(\tau)$ and $|\, u\left(\frac{1+\sqrt{-\frac{205}{5^2}}}{4}\right) = u(\tau p)$. Since h/2 = 4 in Theorem 3.2 we take the 4th power of [9]

[10] $$(G_m G_{\frac{m}{p^2}})^4 = 2^{2/3}(|u(\frac{1+\sqrt{-m}}{4})| * (|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4}))^{-4/3}$$

In Ramanujan's notebook this fourth power is equivalent to the product of fundamental units which in this case (and all cases considered by Berndt) are two sets of radicals. Fortunately, we have enough information from one radical to find the other since we know the value of the LHS of [10]. We are given the values of w_2, h_1, and h_2 in reference Tables [see Berndt] . Using w_2 =2 , h_1=1, and h_2=8 the exponent of the first fundamental unit is 8 and with w_2 =4 , h_1=2, and h_2=1 the power of the unkown unit is 1. If we calculate using equation [10] and the fundamental unit for d_1 = 5, the result is found to be a radical using the root approximation of *Mathematica*.

[11] $$\frac{2^{2/3}(|u(\frac{1+\sqrt{-m}}{4})|*(|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4}))^{-4/3}}{\left(\frac{1+\sqrt{5}}{2}\right)^8} = \frac{1}{2}\left(43+3\sqrt{205}\right)$$

As with most radicals of a number which can be prime factored the RHS of [11] is simplified and the LHS of [10] is equal to the following product,

[12] $$(G_m G_{\frac{m}{p^2}})^4 = (\frac{1+\sqrt{5}}{2})^8 * (\frac{3\sqrt{5}+\sqrt{41}}{2})^2$$

Since we know the value of G_{205} from equation [6], we find the value of the last radical term(s) (LT) in G_{205}

[13] $$\text{LT} = (u(\tau p)\ \ /u(\tau)\ )^{1/6}$$

This appears to be true for all discriminants which require the use of Theorem 3.2 in Berndt's paper. In the next example below we find the alternate value of LT when using Theorem 3.1. However, it is only true if the exponents on the RHS of [12] are corrected for the different powers needed to cancel $G_{\frac{m}{p^2}}$ from [8] and [9] to give G_m. Surprisingly, this number is always the 1/8 th power. It is shown with other class numbers that the number is always 1/h. The ability to use a single factor eliminates the need to calculate any of the values of P and Q which are done in Berndt's first proof of Ramanujan's class invariant. If we take the product of the RHS of [12] to the 1/8th power with LT we indeed have the value of G_{205}.

So it remains to find LT in radical form. This is easily done using the *Mathematica* root approximation of LT. We find

[14] $$\text{LT} = Root\ of\ 1 - 3z^2 - 6z^4 - 3z^6 + z^8 = 0\ = \frac{1}{2}\sqrt{3 + \sqrt{41} + \sqrt{2(17 + 3\sqrt{41})}}$$

[15] $$G_{205} = (\frac{1+\sqrt{5}}{2})^1 * (\frac{3\sqrt{5}+\sqrt{41}}{2})^{1/4} * \frac{1}{2}\sqrt{3 + \sqrt{41} + \sqrt{2(17 + 3\sqrt{41})}}$$

Converting this to the solution of the irreducible polynomial solved by the eta quotient, equation [1] we multiply by $2^{1/4}$. We can now use this radical form to solve for $|u(\tau p)|$ and $|u(\tau)|$. Take the square root of [6] and invert to find $u(\tau)$ then simplify.

[16] $$|u(\tau)| = 2/((\frac{1+\sqrt{5}}{2}) * (\frac{3\sqrt{5}+\sqrt{41}}{2})^{1/4} * 2^{1/4}(\frac{1}{2}\sqrt{3 + \sqrt{41} + \sqrt{2(17 + 3\sqrt{41})}}))^3$$

$$= \frac{128}{(1+\sqrt{5})^3(3\sqrt{5}+\sqrt{41})^{3/4}(3+\sqrt{41}+\sqrt{34+6\sqrt{41}})^{3/2}}$$

It can be shown from [13] that $|u(\tau p)| = \text{LT}^6 * |u(\tau)|$

[17] $$|u(\tau p)| = \frac{2(3+\sqrt{41}+\sqrt{34+6\sqrt{41}})^{3/2}}{(1+\sqrt{5})^3(3\sqrt{5}+\sqrt{41})^{3/4}}$$

One can easily show that the last term in Berndt is equal to LT by taking the 4th power and the squaring each remaining listed radical term. This method has been successful in finding radical forms for other negative discriminants proved by Berndt, (205, 145, 445 and 505) which follow conditions of Theorem 3.2.

Example d = -301

Using $\tau = \frac{1+\sqrt{-301}}{4}$ and $\tau p = \frac{1+\sqrt{-\frac{301}{7^2}}}{4}$ the associated octic q continued fractions are calculated to obtain $|\,u\left(\frac{1+\sqrt{-301}}{4}\right) = u(\tau)$ and $|\,u\left(\frac{1+\sqrt{-\frac{301}{7^2}}}{4}\right) = u(\tau p)$. Since h/2 = 4 in Theorem 3.1 we take the 4th power of [8]

[18] $$\left(\frac{G_m}{G_{\frac{m}{p^2}}}\right)^4 = \left(|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4})|/|u(\frac{1+\sqrt{-m}}{4})|\right)^{4/3}$$

In Ramanujan's notebook this fourth power is equivalent to the product of fundamental units which in this case (and all cases considered by Berndt) are two sets of radicals. A above, we have enough information from one radical to find the other since we know the value of the LHS of [18]. We are given the values of w_2, h_1, and h_2 in reference Tables [see Berndt] . Using w_2 =2 , h_1=1, and h_2=1 the exponent of the first fundamental unit is 1 and with w_2 =4 , h_1=1, and h_2=1 the power of the unkown unit is 1/2. If we calculate using equation [18] and the fundamental unit given above for d_1 = 7, the result is found to be a radical using the root approximation of *Mathematica*.

[19] $$\left(\frac{(|u(\frac{1+\sqrt{-\frac{m}{p^2}}}{4})|/|u(\frac{1+\sqrt{-m}}{4})|)^{4/3}}{\left(\frac{8+3\sqrt{7}}{1}\right)^1}\right)^{1/(\frac{1}{2})} = \frac{1}{2}\left(22745 + 1311\sqrt{301}\right)$$

As with this radical 301 can be prime factored into 7*43 and the RHS of [19] is simplified. The LHS of [18] is equal to the following product,

[20] $$\left(\frac{G_m}{G_{\frac{m}{p^2}}}\right)^4 = \left(\frac{8+3\sqrt{7}}{1}\right)^1 * \left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^1$$

Since we know the value of G_{301} from equation [6], we find the value of the last radical term(s) (LT) in G_{301}

[21] $$\mathrm{LT} = 2^{1/12}(u(\tau) \;\; * u(\tau p) \;)^{-1/6}$$

This appears to be true for all discriminants which require the use of Theorem 3.1 in Berndt's paper. However, it is only true if the exponents on the RHS of [12] are corrected for the different powers needed to cancel $G_{\frac{m}{p^2}}$ from [8] and [9] to give G_m. As above this number is also the 1/8 th power. The ability to use this single factor eliminates the need to calculate any of the values of P and Q which are done in Berndt's first proof of Ramanujan's class invariant. If we take the product of the RHS of [20] with $LT^{(1/8)}$ we indeed have the value of G_{301}.

So it remains to find LT in radical form. This is easily done using the *Mathematica* root approximation of LT. We find

[22] LT = $Root\ of\ [1 - 88z^4 - 169z^8 - 88z^{12} + z^{16} = 0]$

$$= (\tfrac{1}{2}(44 + 7\sqrt{43} + \sqrt{7(577 + 88\sqrt{43})}))^{1/4}$$

which is solvable in radicals. The result is

[23] $$G_{301} = (\frac{8+3\sqrt{7}}{1})^{1/8} * (\frac{3\sqrt{5}+\sqrt{41}}{2})^{1/8} * (\tfrac{1}{2}(44 + 7\sqrt{43} + \sqrt{7(577 + 88\sqrt{43})}))^{1/4}$$

Converting this to the solution of the irreducible polynomial solved by the eta quotient, equation [1] we multiply by $2^{1/4}$. We can now use this radical form to solve for $|u(\tau p)|$ and $|u(\tau)|$. Take the square root of [6] and invert to find $u(\tau)$ then simplify.

[24] $$|u(\tau)| = 2\Big/\left(\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)^{1/8} * \left(\frac{8+3\sqrt{7}}{1}\right)^{1/8} * 2\text{^}(1/4)\left(\left(\left(\frac{1}{2}\left(44 + 7\sqrt{43} + \sqrt{7\left(577 + 88\sqrt{43}\right)}\right)\right)^{1/4}\right)\right)\right)^3$$

$$= \frac{2(16-6\sqrt{7})^{3/8}}{(44+7\sqrt{43}+\sqrt{7(577+88\sqrt{43})})^{3/4}(57\sqrt{7}+23\sqrt{43})^{3/8}}$$

It is shown from [13] that $|u(\tau p)|$ = $LT^6 * |u(\tau)|$

[25] $$|u(\tau p)| = \frac{2(16-6\sqrt{7})^{3/8}}{(44+7\sqrt{43}+\sqrt{7(577+88\sqrt{43})})^{3/4}(57\sqrt{7}+23\sqrt{43})^{3/8}}$$

One can easily show that the last term in Berndt for G_{301} is equal to LT by taking the 8th power and the squaring each remaining listed radical term. This method has been successful in finding radical forms for other negative discriminants proved by Berndt, (65, 69, 77,141, 213, and 445) which follow conditions of Theorem 3.1.

Although h = 8, the genus structure of m = 217 and 553 differ in type, but these discriminants were successfully analyzed using Theorem 3.1. For m = -553 a 4th order equation could be solved to obtain the first two terms in a single radical form and the last term was also a root of a 4th order polynomial!

The conditions of theorems 3.1 and 3.2 may be expanded by letting m = 3 (mod 4). I find the above method applies to m = 95 and 295. Both have class number 8 and are solvable by theorem 3.2. Both G_{95} and G_{295} can also be solved directly by solving for the root of a quadratic equation:

[26] $$(z-|u(\tau)|)\ (z-|u(\tau p)|) = 0$$

The roots z, are solvable by radicals and using the qkQ equation [5] it is easy to show that

[27a,b] $$G_{95} = \frac{2}{(\sqrt{3-3\sqrt{5}+4\sqrt{-1+2\sqrt{5}}}-\sqrt{1+3\sqrt{5}+4\sqrt{-1+2\sqrt{5}}-2\sqrt{2(7+5\sqrt{5})}})^{1/3}}$$

$$G_{295} = \frac{2^{7/12}}{(\sqrt{-16+7\sqrt{5}+\sqrt{-594+266\sqrt{5}}}-\sqrt{-3+6\sqrt{5}-\sqrt{-434+246\sqrt{5}}+\sqrt{-594+266\sqrt{5}}})^{1/3}}$$

A close inspection of the results above and the qkQ equation shows a new invariant. Let G_n be the class invariant for n in which $\tau = \frac{1+\sqrt{-n}}{4}$. Define $u(\tau)$ and $u(\tau p)$ from equation [4] where $\tau p = \frac{1+\sqrt{-\frac{n}{p^2}}}{4}$ and p is a prime such that (p|n). Then for all class numbers,

[28] $$G_n * (|u(\frac{1+\sqrt{-n}}{4})|)^{1/3} = 2^{1/12}$$

A general polynomial in z is solvable in radicals if it is of degree ≤ 4 for all powers of z, or if all powers of z are divisible by an integer A, resulting in a general polynomial of degree ≤ 4. The equations used in the two examples above lead to the following theorem true for all class number h:

Theorem 1 *- A class invariant G_n is solvable in radicals if the product of the moduli from the octic q continued fractions, $(u(\tau) * u(\tau p))^a$ is an algebraic integer from a general polynomial solvable in radicals where a = 1 or a simple power or fraction. This product is either the first or last term in G_n depending on whether the solution depends on equation [8] or equation [9] above, respectively. The remaining term can then be derived from equations [9] and [28].*

In the examples above we find this to be true. Some results are shown in the following table

n	Power "a"/Prime"p"	Root from polynomial	Term
205	2/5	$16 - 725760z + 97208z^2 + 181440z^3 + z^4 = 0$	first
301	1/7	$4 - 2408z - 1544z^2 - 1204z^3 + z^4 = 0$	last
217	1/7	$4 - 752z + 1048z^2 - 376z^3 + z^4 = 0$	last
553	1/7	$4 - 38336z + 48424z^2 - 19168z^3 + z^4 = 0$	last
295	1/5	$-1 + 818z + 468z^2 + 104z^3 + 16z^4 = 0$	first
133	1/7	$45 - 17\sqrt{7}$	first
63	1/7	$\frac{1}{4}(5 - \sqrt{21})$	first
57	1/3	$-13 + 3\sqrt{19}$	last
25	1/5	$4 - 36z^2 + z^4 = 0$	last

From the root radicals it is a straightforward process to find G_n, $|u(\tau)|$ and $|u(\tau p)|$. The following class invariants were found by this method;
$G_{65}, G_{69}, G_{77}, G_{117}, G_{141}, G_{145}, G_{153}, G_{205}, G_{213}, G_{217}, G_{265}, G_{301}, G_{441}, G_{445}, G_{505}, G_{553}$. Although the form of these radicals may not be similar to those obtained by other methods, equivalence of solution can be demonstrated. It remains to show the inverse of this theorem, that no solution in radicals is possible if the moduli product is not a root of a solvable equation. In this case the theorem can be extended to the quotient $(u(\tau)/u(\tau p))^a$ which may be solvable. Equation [18] then applies to solve. This method was required for finding G_{441}. In some problems, such as G_{625} a 5th order equation is found for both product and quotient which are not solvable in radicals unless alternative methods are used (see "Ramanujan's Class Invariant G_{625}", Mathematics Stack Exchange, 2017).

Richard Turk January 26, 2019