The Ramanujan Octave and Discriminants -4m

Ramanujan showed that class invariants are expressed by the eta quotient.

[B1] $$G_m = 2^{-1/4} f(-m) = 2^{-1/4} |n(\tau)/n(\tau 1)|$$

I showed that for odd discriminants the equations are satisfied using the quadratic field $\tau 1 = \frac{1+\sqrt{-m}}{2}$ and $\tau = \frac{1+\sqrt{-m}}{4}$. The numerical result could be expressed in radical form if the modulus of the octic q continued fraction [B2] was calculated and a q cubic solution equation (qkQ), [B2] was developed to correlate G_m with $u(\tau)$.

[B2] $$u(\tau) = \sqrt{2} * q^{1/8} * \prod_{n>1} \frac{(1-q^{2n-1})}{(1-q^{4n-2})^2} = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5 \ldots\ldots}}}}}}$$

Here q = $e^{2\pi i \tau}$.

[B3] $$G_m = ((2/(u(\tau)))^{2/3}/\sqrt{2})^{1/2}$$

The Ramanujan octave was defined from the invariant found from [B3],

[B4] $$G_m * (|u(\tfrac{1+\sqrt{-m}}{4})|)^{1/3} = 2^{1/12}$$

where $2^{1/12}$ is the frequency ratio of the equal temperament 12 tone chromatic scale found in Western music. This system was demonstrated in Chapter 29 for various odd valued negative discriminants. In this appendix the system is extended to include even valued negative discriminants. Unfortunately, the G values of even quadratic fields are calculated from "little" g by equation [B5] which is not directly expressed using [B2] and [B3].

[B5] $$g_m = 2^{-1/4} f_1(-m) = 2^{-1/4} |n(\tau/2)/n(\tau)|$$

with the quadratic field $\tau = \sqrt{-m}$.

It is shown in the literature that u(τ) is an eta modular function and generally a complex number. When m is even $|u(\tau)|$ is a complex number and the conjugate is required in the calculation. Let $|u(\tau)|$ be calculated from equation [B2] then with $\tau = \frac{1+\sqrt{-m}}{4}$, [B6] is equivalent to [B2].

[B6] $$|u(\tau)| = \sqrt{2} * \frac{\eta(\tau)*(\eta(4\tau))^2}{(\eta(2\tau))^3} * Conjugate[\sqrt{2} * \frac{\eta(\tau)*(\eta(4\tau))^2}{(\eta(2\tau))^3}]$$

Since this equation does not give a correct value for g_m when applied to [B3] and [B4] a correction factor is needed. I find that the real value of U as defined in Chapter 28 is this correction.

[B7] $$U = \frac{\eta(\tau)}{(\eta(2\tau))}$$

using a special value of τ.

When calculating G_m we find that fractional values of τ can be used in calculating $u(\tau)$. If we then choose to calculate g_m for an even number, m we can multiply by 4 and let $\tau = \frac{\sqrt{-4m}}{4}$ in [B7]. A new invariant is found like [B4], the Ramanujan octave for even discriminants -4m, where m is an integer if D = -4m = 0 (mod 4) and m is the ratio 4m/4 if D = -4m = 2 (mod 4). (Example D = -4*9 = -36 , m = 9, D = -38, m = 19/2)

[B8] $$g_{4m} * \frac{(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}}{U(\frac{\sqrt{-4m}}{4})} = 2^{1/12}$$

The real ratio $\frac{(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}}{U(\frac{\sqrt{-4m}}{4})}$ has a significant advantage for finding radical forms. As shown in the examples below there is a "resonance of solvability" if this ratio is multiplied or divided by equal temperaments based on $2^{3/12}, 2^{-3/12}$ *or* $2^{1/8}$. The later can be expressed musically by a major scale of 8 notes with $2^{2/8} = 2^{3/12}$ where $2^{3/12}$ is equivalent and increase in frequency of a minor third.

Proposition 1 – The q continued fraction ratio $\frac{(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}}{U(\frac{\sqrt{-4m}}{4})}$ *,is expressible in radicals for negative discriminants -4m = 0 mod 4 and 2 mod 4.*

Proposition 2 – The octahedral equation $\left(\frac{(u(\frac{1+\sqrt{-m}}{4}))}{\left(U(\frac{\sqrt{-4m}}{4})\right)^3}\right)^8 - 1$ *,is a real number expressible in radicals for negative discriminants -4m = 0 mod 4 and 2 mod 4.*

Note that proposition 2 is the real value version of $\left|\left(\frac{(u(\frac{1+\sqrt{-m}}{4}))}{1}\right)^8 - 1\right| = 1$ where $u(\tau)$ is a complex number and m is an integer.

From [B8] let;

[B9] $$|u(\tau)|_e = \frac{|u(\frac{1+\sqrt{-m}}{4})|}{(U(\frac{\sqrt{-4m}}{4})|)^3}$$

[B10] $$g_{-4m} = ((2/(|u(\tau)|_e))^{2/3}/\sqrt{2})^{1/2}$$

The following Table shows some values $|u(\tau)|_e$ for various even discriminants. These radicals are obtained using *Mathematica*; and are shown to give equivalent g_m values to the radical expressions from Weber [1]. Using these values in [B10] results in g_m.

Discriminant -4m	Radical Form of $\|u(\tau)\|_e$
-8	$(-\frac{7}{2} + \frac{5}{\sqrt{2}})^{1/8}$
-10	$2^{1/4}\sqrt{-2+\sqrt{5}}$

-12	$\frac{(-5+3\sqrt{3})^{1/4}}{2^{3/8}}$
-14	$\sqrt{2+2\sqrt{2}-\sqrt{2(5+4\sqrt{2})}}$
-18	$\frac{-2+\sqrt{6}}{2^{1/4}}$
-26	$\frac{(\frac{1}{3}(8+(359-12\sqrt{78})^{1/3}+(359+12\sqrt{78})^{1/3})-\sqrt{-4+\frac{1}{9}(8+(359-12\sqrt{78})^{1/3}+(359+12\sqrt{78})^{1/}}}{\sqrt{2}}$
-30	$\sqrt{10-3\sqrt{10}-2\sqrt{2(19-6\sqrt{10})}}$
-34	$2^{1/4}\sqrt{9+2\sqrt{17}-2\sqrt{37+9\sqrt{17}}}$
-48	$\frac{(-16+7\sqrt{6}-\sqrt{6(59-24\sqrt{6})})^{3/8}}{2^{3/4}}$
-60	$\frac{(305+141\sqrt{5}-3\sqrt{21378+9558\sqrt{5}})^{1/4}}{2^{7/8}}$
-130	$2^{1/4}\sqrt{684+85\sqrt{65}-2\sqrt{234370+29070\sqrt{65}}}$
-190	$2^{1/4}(6+5\sqrt{2}-\sqrt{85+60\sqrt{2}})^{3/2}$

Note in many of these radicals the Ramanujan octave is part of the expression. In most cases the value $2^{1/4}$ or $2^{n/8}$ occurs.

As the first example, consider discriminant -22 = 2 mod 4 with m = 11/2. Let $\tau = \frac{1+\sqrt{-11/2}}{4}$ and calculate $\left|u(\frac{1+\sqrt{-11/2}}{4})\right|$ from the octic q continued fraction. Next use $\tau = \frac{\sqrt{-22}}{4}$ in [B7] to find U with the *Mathematica* DedekindEta function. The ratio [B9] is then easily calculated and the result is an algebraic integer of a radical root to the polynomial $2-20z^2+z^4=0$. In this instance a fourth order polynomial is found. An equivalent polynomial with smaller coefficients is found if the ratio is divided by $2^{1/4}$. In this case the polynomial $-1+14z^2+z^4=0$ is also solvable. As in many of these examples, multiple forms of the radical expressions can be obtained. One can choose between two equivalent forms of $|u(\tau)|_e$;

[B10a,b] $$|u(\tau)|_e = \sqrt{10-7\sqrt{2}} \text{ or } 2^{1/4}\sqrt{-7+5\sqrt{2}}$$

Substituting into [B10] I obtain

[B11a,b] $$g_{-22} = \frac{2^{1/12}}{(10-7\sqrt{2})^{1/6}} \text{ or } \frac{1}{(-7+5\sqrt{2})^{1/6}}$$

In the second example the method above demonstrates several ways to obtain a radical form based on propositions 1 and 2.

As the second example, consider discriminant -24 = 0 mod 4 with m = 12. Let $\tau = \frac{1+\sqrt{-12}}{4}$ and calculate $\left|u(\frac{1+\sqrt{-12}}{4})\right|$ from the octic q continued fraction. Next use $\tau = \frac{\sqrt{-24}}{4}$ in [B7] to find U with the *Mathematica* **DedekindEta** function. The ratio [B9] is then easily calculated but unlike the example above, the result is not an algebraic integer of a solvable polynomial. However, based on proposition 2 the octahedral equation results in a root of a solvable polynomial. If we solve the polynomial $-9791 - 1592z + 13704z^2 - 2336z^3 + 16z^4 = 0$, a value for $|u(\tau)|_e$ is found.

[B12] $$|u(\tau)|_e = (\tfrac{1}{2}(-71 + 42\sqrt{3} - 3\sqrt{998 - 576\sqrt{3}}))^{1/8}$$

It would be preferable to find a monic polynomial to verify this solution. If we apply a Ramanujan octave by dividing $|u(\tau)|$ by $2^{-1/8}$ and take the 8th power as suggest by [B12] one obtains a algebraic number which is the root to the monic polynomial $1 - 9436z + 1698z^2 + 284z^3 + z^4 = 0$. As anticipated, many of the coefficients are also reduced. As above and in many cases shown in the Table above, use of the Ramanujan octave simplifies the polynomial and targets a minimal polynomial of the discriminant. The equations however can also lead to the same result and [B12] is verified by using the radical form of the monic polynomial. Substituting into [B10],

[B13] $$g_{-24} = \frac{2^{1/8}}{(-71+42\sqrt{3}-3\sqrt{998-576\sqrt{3}})^{1/24}}$$

The solution can be verified by [B8],

[B14] $$g_{-24} * (|u(\tau)|_e)^{1/3} = 2^{1/12}$$

Using the **MinimalPolynomial** function on *Mathematica* the minimal polynomial is 96th order but solvable,

[B15] $$4096 + 145408x^{24} + 108672x^{48} - 75488x^{72} + x^{96}$$

Many of the minimal polynomials for g_{-4m} are found to be powers of 4, 8 and 12.

In all the examples above the values of g_{-4m} are calculated to 500 places and shown to be equivalent to the values published by Weber.

The Complete Calculation with the q Octic Continued Fractions

Since the topics of the last few chapters are using Ramanujan's q continued fraction to find class invariants, we seek a solution for discriminants -4m using these fractions. This requires us to replace the calculation of U based on the Weber function [B7] with another expression. Fortunately, the answer is directly available to us.

In place of calculating $U(\frac{\sqrt{-4m}}{4})$ we can calculate $u(\frac{\sqrt{-4m}}{4})$ from [B2] providing us with two numbers from q octic continued fractions. The answer again appears from the invariant [B14],

[B16] $$(1/g_{-4m}) * \frac{|u(\frac{1+\sqrt{-m}}{4})|^{1/3}}{u(\frac{\sqrt{-4m}}{4})} = 2^{1/12}$$

Rearranging this invariant provides a direct calculation of g_{-4m}! Note that the upper value $|\mathrm{u}(\frac{1+\sqrt{-m}}{4})|^{1/3}$ is the modulus of a complex number and the lower value $\mathrm{u}(\frac{\sqrt{-4m}}{4})$ is a real number but the Ramanujan octave is again discovered in this simple equation without requiring modular eta functions or quotients. Using this simple expression is a convenient method to directly find g_{-4m}. In many cases *Mathematica* can find the minimal polynomial if the expression is squared. For example, calculating g_{-100} gives,

[B17] $$(g_{-100}) = \frac{|\mathrm{u}(\frac{1+\sqrt{-25}}{4})|^{1/3}}{2^{1/12} * \mathrm{u}(\frac{\sqrt{-100}}{4})}$$

which does not produce a root approximation. However, if this number is squared it is found to be the root of the polynomial $16 - 752z^4 + 204z^8 - 8828z^{12} + z^{16}$ which is related to the minimal polynomial of the class invariant, $16 - 752z^8 + 204z^{16} - 8828z^{24} + z^{32}$.

In conclusion, two invariants have been found to provide a calculation of all class invariants. Applying [B4] with odd discriminants and [B16] with even discriminants should help in finding values and radical forms for these class invariants. Moreover, these expressions illustrate a direct relationship between G_{-m}, g_{-4m} and the octic continued fraction

[B18] $$(g_{-4m} * G_{-m}) = \frac{1}{\mathrm{u}(\frac{\sqrt{-4m}}{4})}$$

where m is an odd number!

The numbers 8 and 12 appear in the elements of the octahedron. The platonic solid has 8 faces and 12 edges and 6 vertices. The solid may have a natural relation to the 12-tone scale only based on the number 12 but it also extends into the mathematics of modular functions. The powers 1/12 and 1/24 are found in many modular equations such as the j-invariant where 12^3 is a factor used to impose independence of the invariant for elliptic curves in any coordinate system. The powers 12, 24 and 8 appear in the Weber function relations to the j-invariant. Although other variants of the Ramanujan octave have appeared in the literature, the results discussed in this paper do not have any significance beyond these mathematical observations but still serves as an intriguing connection of mathematics to architectural geometry and music harmony.

1. H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.

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