The Ramanujan Octave and Examples of the Existence Theorem

In this appendix I will outline examples of class invariants calculated using the final *Theorem 1* from the previous chapter which is shown below. I will first go through the example for m = 205 discussed in that chapter and then choose an example from a class number of 2 using m = 117 to illustrate how the theorem is applied to other class numbers. It should be mentioned that the theorem presently only applies to discriminants 1 or 3 (mod 4) where radical existence can be shown. I allude to the example from Chapter 28 for m = 59 where the class invariant can be determined since a solvable polynomial is found. At the end of this appendix I will expand on the new invariant found in equation [28] and [A2] and demonstrate its analogy to the octave in music.

Theorem 1 (Existence of Solvability)- *A class invariant G_n is solvable in radicals if the product or quotient of the moduli from the octic q continued fractions, $(u(\tau) * u(\tau p))^a$ or $(u(\tau)/u(\tau p))^a$ is an algebraic integer from a general polynomial solvable in radicals where a = 1 or a simple power or fraction. This product is either the first or last term in G_n depending on whether the solution depends on a polynomial solution of the first or last term for G_n. The remaining term can then be derived from a polynomial for the remaining term.*

Example 1: m = 205, Class number h = 8

We start by using $\tau = \frac{1+\sqrt{-205}}{4}$ and since 41|m, $\tau p = \frac{1+\sqrt{-\frac{205}{41^2}}}{4}$ and the associated octic q continued fractions are calculated using *Mathematica* to obtain $\left|u\left(\frac{1+\sqrt{-205}}{4}\right)\right| = u(\tau)$ and $\left|u\left(\frac{1+\sqrt{-\frac{205}{41^2}}}{4}\right)\right| = u(\tau p)$. Both $(u(\tau) * u(\tau p))^a$ and $(u(\tau)/u(\tau p))^a$ are calculated and it is found by root approximation that both are algebraic integers expressed as 4th order polynomials showing existence of a radical solution for a = 1. The later ratio is used since it has the smallest coefficients. We find the root of

$z^4 - 90z^3 + 18z^2 - 90z - 1 = 0$. Following equation [13] in Chapter 29 we immediately can calculate the last term (for $(u(\tau)/u(\tau p))$).

[A1] $$\mathrm{LT} = (u(\tau)\ /u(\tau p)\ )^{-1/6} = \left(\frac{45}{2} + \frac{7\sqrt{41}}{2} - \sqrt{\frac{5}{2}\left(403 + 63\sqrt{41}\right)}\ \right)^{-1/6}$$

Next equations are used to find the numerical values of G_{205} and the first term. The first is the invariant.

[A2] $$G_n * \left(\left|u\left(\frac{1+\sqrt{-n}}{4}\right)\right|\right)^{1/3} = 2^{1/12}$$

[A3] $$G_{205} = \left((2/(u(\tau)))^{\mathrm{k1}/3}/\mathrm{Q1}\right)^{1/2}$$

[A4] $$\mathrm{FT} = \left(2^{1/3}(u(\tau) * u(\tau p))^{-2/3}\right)^{h/4}$$

where k1 = 2, Q1 = $\sqrt{2}$, and h = 8. These equations are derived from the invariant equation [A2] which is true for both $u(\tau)$ *and* $u(\tau p)$ and the product $u(\tau) * u(\tau p)$. Whenever the last term is calculated first from the octic ratio as in [A1] both equations [A3] and [A4] apply. The numerical accuracy of these calculations can be shown through a simple check.

[A5] $$G_{205} = \mathrm{FT}^x * \mathrm{LT}$$

with x being the power 1/h if the right and left agree. To check this, take the log of the equation and solve for x

[A6] $$x = 1/h = \text{Log}[G_{205}/LT]/\text{Log}[FT]$$

and we find x = 1/8 as expected. The first term FT^x can be calculated from the ratio G_{205}/LT. I find that this number is an algebraic integer, a root of the polynomial $z^{16} - 45z^{12} + 2z^8 + 45z^4 + 1 = 0$ which is solvable by radicals.

[A7] $$FT^x = \frac{(45+21\sqrt{5}+\sqrt{82(47+21\sqrt{5})}\,)^{1/4}}{\sqrt{2}}$$

Put the first and last term together and prior to simplification get G_{205},

[A8] $$G_{205} = \frac{(45+21\sqrt{5}+\sqrt{82(47+21\sqrt{5})})^{1/4}}{\sqrt{2}}) * (\frac{45}{2} + \frac{7\sqrt{41}}{2} - \sqrt{\frac{5}{2}(403 + 63\sqrt{41})})^{-1/6}$$

Using equation [A2], a radical form of $u(\tau)$ can be obtained. From [A1] a radical form of $u(\tau p) = LT^6 * u(\tau)$ is found.

It is easily shown both numerically and by power expansion that Berndt's solution is equivalent since my first and last term, (LHS) below match his 3 terms, (RHS);

[A9] $$(\frac{45}{2} + \frac{7\sqrt{41}}{2} - \sqrt{\frac{5}{2}(403 + 63\sqrt{41})})^{-1/6} = (\sqrt{(\frac{7+\sqrt{41}}{8})} + \sqrt{(\frac{\sqrt{41}-1}{8})})$$

[A10] $$\left(\frac{\left(45+21\sqrt{5}+\sqrt{82(47+21\sqrt{5})}\right)^{1/4}}{\sqrt{2}}\right) = (\frac{1+\sqrt{5}}{2}) * (\frac{3\sqrt{5}+\sqrt{41}}{2})^{1/4}$$

Example 2: m = 117, Class number h = 2

We start by using $\tau = \frac{1+\sqrt{-117}}{4}$ and since 13|m, $\tau p = \frac{1+\sqrt{-\frac{205}{13^2}}}{4}$ and the associated octic q continued fractions are calculated using *Mathematica* to obtain $|u\left(\frac{1+\sqrt{-117}}{4}\right)| = u(\tau)$ and $|u\left(\frac{1+\sqrt{-\frac{117}{13^2}}}{4}\right)| = u(\tau p)$. Both $(u(\tau) * u(\tau p))^a$ and $(u(\tau)/u(\tau p))^a$ are calculated and it is found by root approximation that both are algebraic integers expressed as 4th order polynomials showing existence of a radical solution for a = 1. In this example, I choose the former to show the procedure for starting with the product. We find the root of $z^8 + 3600z^6 - 4808z^4 - 14400z^2 - 16 = 0$. We immediately can calculate the first term using $(u(\tau) * u(\tau p))$ and equation [A4] above and h = 2

[A11] $$FT = (2^{1/3}(u(\tau) * u(\tau p))^{-2/3})^{h/4} = \frac{1}{(-450+125\sqrt{13}-4\sqrt{39(649-180\sqrt{13})})^{1/6}}$$

Next equations are used to find the numerical values of G_{117} and the last term. The first is the invariant.

[A12] $G_n * (|u(\frac{1+\sqrt{-n}}{4})|)^{1/3} = 2^{1/12}$

[A13] $G_{117} = ((2/(u(\tau)))^{k1/3}/Q1)^{1/2}$

[A14] $LT = (u(\tau p)\ /u(\tau)\ )^{1/6}$

where k1 = 2 and Q1 = $\sqrt{2}$. These equations are derived from the invariant equation [A12] which is true for both $u(\tau)$ *and* $u(\tau p)$ and the quotient $u(\tau p)\ /u(\tau)$. Whenever the first term is calculated initially from the octic ratio as in [A11] both equations [A13] and [A14] apply. The numerical accuracy of these calculations can be shown through a simple check.

[A15] $G_{117} = FT^x * LT$

with x being the power 1/h if the right and left agree. To check this, take the log of the equation and solve for x

[A16] $x = 1/h = Log[G_{117}/LT]/Log[FT]$

and we find x = 1/2 as expected. The last term LT can be calculated from [A14]. I find that this number is an algebraic integer, a root of the polynomial $z^8 - 4z^6 + 3z^4 - 4z^2 + 1 = 0$ which is solvable by radicals.

[A17] $LT = \sqrt{\frac{1}{2}(2+\sqrt{3}+\sqrt{3+4\sqrt{3}})}$

Put the first and last term together remembering to take the 1/h power of the first term to get G_{117},

[A18] $G_{117} = (\frac{1}{(-450+125\sqrt{13}-4\sqrt{39(649-180\sqrt{13})})^{1/6}})^{1/2} * \sqrt{\frac{1}{2}(2+\sqrt{3}+\sqrt{3+4\sqrt{3}})}$

Using equation [A12], a radical form of $u(\tau)$ can be obtained. From [A14] a radical form of $u(\tau p) = LT^6 * u(\tau)$ is also found.

A solution in radical form is not available in the literature however the value can be checked against the Weber function f[-m].

[A19] $f[-117] = (\text{DedekindEta}[\frac{1}{4}(1+3i\sqrt{13})]/\text{DedekindEta}[\frac{1}{2}(1+3i\sqrt{13})])$

Multiplying by the conjugate of [A19] and multiplying by $2^{-1/4}$ we obtain the same value for G_{117} checked to 500 places on *Mathematica*.

Ramanujan's Octave

The octic q continued fraction mentioned in Chapter 28 shows an interesting relation to the musical octave. Although 8 is the number of notes in a musical scale which ranges one octave, e.g. middle C to C1 one octave higher, or G to G1, the invariant in equation [A2] represents a 12 tone well-tempered scale, a chromatic scale of 12 notes ranging 1 octave. This equal temperament divides the octave into 12 parts such that each raised note is a fixed ratio higher in frequency. This applies to tempered strings on a piano or violin as well as a tempered wind or reed instrument. This ratio is $K = 2^{1/12}$ such that powers of K represent intervals on the scale. For example, intervals of a fourth and fifth are K^5 and K^7 respectively. Similarly, we can go down a scale by taking negative powers. If we tune a violin to the note A at 440 cps then middle C is K^{-9} or 261.63 cps. A fourth above C is then $K^5*261.63 = 349.23$ cps.

We can take the twelfth power of K and find $K^{12} = 2$. Taking the 12^{th} power of [A2] we find

[A20] $$G_m^{12}|u(\tau)|^4 \;=\; 2$$

Looking back at Chapter 28 I defined |U| and |U2| as

[A21] $$|U| = \frac{\eta(\tau)}{(\eta(2\tau))^1} * \text{ Conjugate } \left(\frac{\eta(\tau)}{(\eta(2\tau))^1}\right)$$

[A21b] $$|U2| = \sqrt{2} * \frac{(\eta(4\tau))^2}{(\eta(2\tau))^2} * \text{ Conjugate } \left(\sqrt{2} * \frac{(\eta(4\tau))^2}{(\eta(2\tau))^2}\right)$$

[A21c] $$|U|^4 * |U2| = 2$$

From [A19] we find that |U| = $f[-m]$ = $2^{1/4}$*G_m. Then it can be shown that

[A22] $$|U2| = |u(\tau)|/f[-m]$$

then |U| represents the equal temperament scale of U_m^n with n = integer from 0 to 12 where $U_m^0 = 1$ and $U_m^1 = |U|$.

Let U2 = $\frac{\eta(\tau)}{(\eta(2\tau))^1}$ and $u(\tau)$ be complex numbers. Then

[A23] $$\text{magnitude } [\,(u(\tau)/\text{U2})^n] = U_m^{n/2}$$

Previously, we calculated, magnitude [$(u(\tau)^8 - 1] = 1$ showing a relation to the 8 faced octahedron.

From equations [A20] and A[21c] we see that the class invariant G_m and the invariant $G_m^1 * |u(\tau)|^{1/3}$ define a tonal scale from the tonic root to the 12^{th} tone octave.

As $|u(\tau)|^n$ changes with increasing n the value G_m increases as a product K^n. For example, a value of $|u(\tau)|^n$ corresponds to the class invariant increasing as

[A24] $$G_{m'} = G_m^{\,n} * K^{(1-n)}$$

This equation applies to fractional values of n suggesting that increasing $|u(\tau)|^n$ by fractional powers will correspond to values of $G_{m'}$ for other values of m. Currently no relationship has been found for the class number 8 discriminants.

Another relation involving K is between powers of f[-m] and G_m. Let j be the power of G_m then we have as defined above, the class invariants

[A25] $$f[-m]^j = K^{3j} * G_m^{\,j}$$

Mathematically, the harmonics of the Ramanujan octave are intriguing and require further research into its application to modular equations.

Equation [A24] can be useful with discriminants that are factored with squared primes. If $p^2|m$ then if the remainder is an integer 1 or 3 (mod 4) it is possible to use [A24] to find its class invariant. For example, m = 441 = 5^2 * 49. Calculating $u(\tau p) = \left|u\left(\frac{1+\sqrt{-\frac{441}{5^2}}}{4}\right)\right|$ and $u(\tau) = \left|u\left(\frac{1+\sqrt{-441}}{4}\right)\right|$ the frequency factor n is calculated as

[A26] $n = Log[|u(\tau)|]/Log[|u(\tau p)|]$

then we can calculate G_{49} by inverting [A24] and using [A12] to find G_{441};

[A27] $G_{49} = (G_{441} / K^{(1-n)})^{1/n}$

The resulting algebraic integer is a root to $z^8 - 4z^6 - z^4 - 4z^2 + 1 = 0$ which has a radical solution,

[A28] $G_{49} = \sqrt{\frac{1}{2}(2+\sqrt{7}+\sqrt{7+4\sqrt{7}})}$

This process can be used to find remainders of m only through calculation of $|u(\tau p)|$ and $|u(\tau)|$; provided a solvable polynomial is found. The frequency factor n gives a value of a harmonic frequency much like the calculation of frequencies of equal temperament in the octave. The difference is that n is not usually an integer, but the G_m values are factors based on the factorization of the number m. Within the octave there are natural frequencies that require several constraints. Equation [A2] is a constraint on G_m for **all** the factors of m. It is also important to note that the octahedral platonic solid constraint [A29] is satisfied for all $u(\tau)$ *and* $u(\tau p)$, where p is a factor $p_i|m$ and generates frequency factors n_1, n_2, n_3, … over all divisors of m.

[A29] $\text{Magnitude}\ [\ (u(\tau)^8 - 1) * (u(\tau p_1)^8 - 1) * (u(\tau p_2)^8 - 1) \ldots * (u(\tau p_n)^8 - 1) = 1]$

As in the above calculation of G_{49} we can seek all divisors of m and target the class invariant for these individual numbers. Consider the divisors of m = 2025: $\{1,3,5,9,15,25,27,45,75,81,135,225,405,675,2025\}$. Since $9*225 = 3^2 15^2 = 2025$, a frequency factor can be determined for $u(\tau p1) = \left|u\left(\frac{1+\sqrt{-\frac{2025}{3^2}}}{4}\right)\right|$ and $u(\tau p2) = \left|u\left(\frac{1+\sqrt{-\frac{2025}{9^2}}}{4}\right)\right|$. From [A26], n is calculated by a log ratio of $u(\tau p2)/u(\tau p1)$ and [A24] is of the form,

[A30] $G_{25} = G_{225}^{(n+k)} * K^{(1-n)}$

If k = 0 then we have G_{25} directly. If k > 0 then we have products of $G_{25}\, G_{225}^{k}$. Certain values of k are searched to find solvable polynomials. Once the product is identified and the radical form is found one divides by the known radical form of G_{25} and then takes the (1/k) power to find G_{225}. In this example k = 6 resulting in,

[A31] $G_{225} = \left(\frac{18698+4828\sqrt{15}+\sqrt{699194295+180531192\sqrt{15}}+\sqrt{4+(18698+4828\sqrt{15}+\sqrt{699194295+180531192\sqrt{15}})^2}}{1+\sqrt{5}}\right)^{1/6}$

In essence we have found a class invariant from Ramanujan's octave A[2]! The invariant is the semitone K,

[A32] $G_{225} * \left(\left|u\left(\frac{1+\sqrt{-\frac{2025}{3^2}}}{4}\right)\right|\right)^{1/3} = 2^{1/12} = \mathrm{K}$

There are 15 relations like [A32] for m = 2025. Each invariant is also a semitone of an infinite number of m's. An infinite number of semitones create the Ramanujan octave.