The Ramanujan Octave, Semitones, Chords and Harmonics

The Ramanujan octic q continued fraction has a natural connection to music harmony through its association with class invariants. In previous chalkboards I derived several equations based on the semitones of the chromatic scale of 12 notes of equal temperament. The first semitone connects the class invariant of discriminant -m, G_{-m} to the modulus of the octic q continued fraction $|u(\tau)|$ with $\tau = \frac{1+\sqrt{-m}}{4}$,

[1] $$G_{-m} * (|u(\tfrac{1+\sqrt{-m}}{4})|)^{1/3} = 2^{1/12}$$

where $2^{1/12}$ is the vibrational frequency ratio of an equal tempered 12 tone chromatic scale.

In references (1) a connection of the q continued fractions to modular functions and the octahedron is also discussed. The octic q continued fraction in [1] associates with the octahedron which is a platonic solid having 12- edges, 6 - (vertices) and 8 -faces. It is noted that $u(\tau)^8$-1 times its conjugate is exactly equal to 1. The q continued fraction is said to map points τ on the upper half of the complex plane to a point $u(\tau)$ on an octahedron projected on the complex plane. Symmetry properties of the octahedron projected (mapped) onto a sphere are then preserved when projected on the complex plane. In addition, invariant properties such as the j-invariant of a polynomial with discriminant -m are also preserved on the octahedron. The octahedral equation [2] illustrates this symmetry;

[2] $$(u(\tau)^{16} + 14 * u(\tau)^8 + 1)^3 - (2^{-4}) * \mathrm{j}(\tau) * (u(\tau)^8 * (u(\tau)^8 - 1)^4) = 0$$

This equation is true for all discriminants and is only true for the complex values of $u(\tau)$. Rearranging [2], the j invariant is a ratio of two terms which are also equations of the octahedron defining its edges and vertices.

Further examination of these equations also shows a connection of different class invariants of discriminants -m and -4m. This connection is found between odd values of -m and its even counterpart -4m as well as between even -m and even counterpart -4m. These relations are shown in [3] and [4] respectively,

[3] $$(g_{-4m} * G_{-m}) = \frac{1}{\mathrm{u}(\frac{\sqrt{-4m}}{4})}$$

[4] $$(g_{-4m} * g_{-m}) = \frac{\mathrm{u}(\frac{1+\sqrt{-4m}}{4})\mathrm{u}(\frac{1+\sqrt{-m}}{4})}{2^{1/6}\ \mathrm{u}(\frac{\sqrt{-4m}}{4})\mathrm{u}(\frac{\sqrt{-m}}{4})}$$

where g refers to even values of m and G is used if m is odd.

From [4] it can be easily shown that the invariant for the product is equal to two semitones.

[5] $$\frac{\mathrm{u}(\frac{1+\sqrt{-4m}}{4})\mathrm{u}(\frac{1+\sqrt{-m}}{4})}{(g_{-4m}*g_{-m})\mathrm{u}(\frac{\sqrt{-4m}}{4})\mathrm{u}(\frac{\sqrt{-m}}{4})} = 2^{2/12} = 2^{1/6}$$

Harmonically, two semitones are a major second and 4 semitones a major third, both part of a major harmonic scale in Western music.

The solvability of these equations is possible if either g and G are solvable or the product of g*G is a root of an equation of order <5 or expressed as a polynomial in radical coefficients of order <5. In many cases these invariants can then be expressed neatly in radical form, otherwise the value of these products is known from the RHS of [3] and [4].

From [3] and [4] a layering or chord structure is possible for a given value of m. Let m_1 = odd integer and m_2 = even integer. Then form the structures,

[6a] $$(g_{-16m1} * g_{-4m1})/(g_{-4m1} * G_{-m1})$$

[6b] $$(g_{-16m2} * g_{-4m2})/(g_{-4m2} * g_{-m2})$$

Newforms are then created as

[7a] $$(g_{-64m1} * g_{-16m1})/(g_{-16m1} * g_{-4m1})(g_{-16m1} * g_{-4m1})/(g_{-4m1} * G_{-m1})$$

[7b] $$(g_{-64m2} * g_{-16m2})/(g_{-16m2} * g_{-4m2})(g_{-16m2} * g_{-4m2})/(g_{-4m2} * g_{-m2})$$

These structures show a nullification of invariants of -4m and -16m leading to ratios for $g_{-4^n m}/g_{-m}$ or $g_{-4^n m}/G_{-m}$.

For many values of m these ratios are solvable and lead to radical forms for $g_{-4^n m}$ using known radical forms for (n-1), (n-2)…..1.

Example: Find g_{-192} starting with equation [6b] or [7a].

The values of $(g_{-192} * g_{-48})/(g_{-48} * g_{-12})$ and $(g_{-192} * g_{-48})/(g_{-48} * g_{-12})(g_{-48} * g_{-12})/(g_{-12} * G_{-3})$ are readily calculated from the octic fractions in [3] and [4]. Form the harmonic powers by dividing each by $2^{1/8}$ and taking to the 8th power to find these products are roots to the following equations of 8th order!

Respectively,

$$1 + 6656z + 3904z^2 + 5475584z^3 + 13705264z^4 - 25590784z^5 + 11016064z^6 - 426496z^7 + 16z^8 = 0$$

$$16 + 1024z + 2986624z^2 + 101410816z^3 + 397256944z^4 + 522657664z^5 + 230634304z^6 - 5936384z^7 + z^8 = 0$$

Both these equations are solved using *Mathematica* by solving a second order polynomial of two real coefficients that are expressed in radical form from a 4th order polynomial. In the first case take the 1/8th power and multiply by the radical form of g_{-12} and $2^{1/8}$ to find g_{-192} (see [8]). In the second case, take the 1/8th power and multiply by the radical form of $G_{-3}*g_{-12}/g_{-48}$ and $2^{1/8}$ to find g_{-192} (see [9]). The two solutions for g_{-192} are shown below [a],

[8]

$$2^{1/24}(1+\sqrt{3})^{1/4}(8(833+340\sqrt{6}+12\sqrt{9602+3920\sqrt{6}})+\sqrt{2(4+\sqrt{42-15\sqrt{3}})+64(833+340\sqrt{6}+12\sqrt{9602+3920\sqrt{6}})^2})^{1/8}$$

[9]

$$\frac{1}{(-5+3\sqrt{3})^{1/12}}(-26672+10890\sqrt{6}-30\sqrt{1572690-642048\sqrt{6}})^{1/24}(8(185512+75735\sqrt{6}+\sqrt{68827561698+28098734400\sqrt{6}})$$
$$+2\sqrt{(}49712+20295\sqrt{6}+\sqrt{6(823453883+336173640\sqrt{6})}$$
$$+16(185512+75735\sqrt{6}+\sqrt{68827561698+28098734400\sqrt{6}})^2))^{1/8}$$

Both radical forms are equivalent to 1000 places and demonstrate how divisors of 192 are incorporated into the radical. Without the use of [6] or [7] it would be difficult to find these radical forms. I find that all chord structures that are solvable lead to 8th order polynomials. The equations also allow for fractional values of m. For example, a structure such as $\left(g_{-104} * g_{-26}\right)/\left(g_{-26} * G_{-13/2}\right)$ could be used to find g_{-104} if $G_{-13/2}$ is solvable in radicals.

For even values of -m equation [1] is replaced by

[10] $$g_{-m} * \frac{u(\frac{\sqrt{-4m}}{4})^1}{|(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}} = 2^{-1/12}$$

illustrating the connection of odd and even discriminants to invariants and the semitone $2^{1/12}$. As we increase m from odd to even, the harmonic ratio of their product cancels as

[11] $$g_{-m} * G_{-(m+1)} * \left(\frac{u(\frac{\sqrt{-4m}}{4})}{|(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}}(|u(\frac{1+\sqrt{-(4m+1)}}{4})|)^{1/3}\right) = 1$$

for m even,

[12] $$G_{-m} * g_{-(m+1)} * \left(\frac{u(\frac{\sqrt{-4m}}{4})}{|(|u(\frac{1+\sqrt{-m}}{4})|)^{1/3}}(|u(\frac{1+\sqrt{-(4m+1)}}{4})|)^{1/3}\right) = 1$$

for m odd and applicable for all class invariants m and m+1. We find the product of two consecutive class invariants is the inverse of products of their octic q continued fractions!

A curious calculation has been observed between the octic continued fraction and the morphic numbers which are the real solutions to the cubic equations

[13] $z^3 - z - 1 = 0$ and $z^3 - z^2 - 1 = 0$ with discriminants -23 and -31 respectively.

These solutions can be found **directly** from the octic continued fraction from the inverse $1/|(|u(\frac{n+\sqrt{-m}}{8})|)$. Here for discriminant m = -23, n = +/- 1 (mod8) and for discriminant m = -31, n = +/- 3 (mod8).

Another comparison can be made between equation [1] above and the volume and area of the octahedron. Let edge = length of an edge of an octahedron and V the volume and A its area. Then V = $\frac{\sqrt{2}}{3}(edge)^3$ and A = $2\sqrt{3}(edge)^2$. Solving for edge in terms of V and A and equating results in:

[14] $$(2)^{1/3} * \frac{(3)^{7/12}(V)^{1/3}}{\sqrt{A}} = 1$$

Substituting the equation for area, A and simplifying, results in an equation with V and edge which is like equation [1]

[15] $$\frac{(3)^{1/3}(V)^{1/3}}{\text{edge}} = (2)^{1/6}$$

Taking the square root of both sides we obtain the following equivalence:

[16] $$\frac{(3)^{1/6}(V)^{1/6}}{(edge)^{1/2}} = \frac{(3)^{1/6}((V^{1/2})^{1/3}}{(edge)^{1/2}} = (2)^{1/12}$$

Comparing with equation [1] let

[17a,b] $$V^{1/2} = |u| \quad \text{and} \quad G = \frac{(3)^{1/6}}{(edge)^{1/2}}$$

In this form equations [1] and [16] appear similar for any edge length and volume. Given any radical form of 'edge' results in unique values of G, |u| and V!

The Ramanujan Ladder

We can now go up the scale of class invariants of discriminant -m by using the 'Ramanujan ladder'. The equation for the volume of the octahedron as a function of the cube of the edge (all edges of equal scale) and the Ramanujan octave provide us the tools for constructing the ladder. There are two types of 'rungs' for this ladder, an even and an odd one.

From [17b] the fourth power of G is found to be an inverse square law of the edge.

[18] $$G^4 = \frac{(3)^{2/3}}{(edge)^2}$$

It can be shown from the equation for the volume V and [17a] that the value of $(edge)^2$ is a function of $|u|$:

[19] $$(edge)^2 = \left(\frac{2}{9}\right)^{-1/3} |u|^{4/3}$$

Once |u| is calculated from the octic continued fraction the value of $(edge)^2$ is found and substituted into [18] to find G^4. The Ramanujan ladder is:

[20] $$\frac{G_2^4}{G_1^4} = \frac{edge_1^2}{edge_2^2}$$

From which any G_2 can be found if one value of G_1 is known and both edges are obtained from [19]. The rungs of the ladder are values of |u|:

[21a, b] For m odd: $|u| = |u(\frac{1+\sqrt{-m}}{4})|$ For m even: $|u| = \sqrt{2} * \left(|u(\frac{\sqrt{-m}}{4})|/|u(\frac{1+\sqrt{-m/4}}{4})|^{1/3}\right)^3$

Equation [20] is applied for any combination of odd, even and even/odd values of G_2 and G_1. The rungs on the Ramanujan ladder are unbounded with edges approaching zero and the class invariant approaching infinity!.

1. W. Duke, Continued Fractions and Modular Functions, **Bulletin of the American Mathematical Society**, **42** (2005), 137-162.

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a. The *Mathematica* code for solution to g_{-192} is available on request.