Elliptic Functions and the Ramanujan Octave

The Ramanujan octic q continued fraction has been shown to have a natural connection with class invariants. In this Chapter we remove the restrictions of modular equations involving complex variables and exponential functions which have connected the octic fraction with a class invariant. By introducing Jacobi elliptic integral and elliptic functions this connection is made with only one constant, the k modulus. The elliptic integral was developed by Legendre and Jacobi and has been applied to the motion of pendulums, elliptical coordinates and elliptical orbits in classical mechanics. Elliptic integrals are integrals of rational functions of the form[1],

[1] $$I = \int R(x,y)dx$$

where R is a rational function of two variables with y an algebraic function of x. For certain algebraic curves particularly of genus 1, it was proved in the 19th century that y^2 is a polynomial in x of degree three or four. I have shown that polynomials of degree three and four are solvable in radicals.

[2] $$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$$

Any elliptic integral described by [1] and [2] can be reduced to elliptic integrals of the first, second and third kind. This chapter only considers the elliptic integral of the first kind in Legendre's form

[3] $$F(\phi,k) = \int_0^{\phi}(1-k^2 sin^2 t)^{-1/2}dt$$

where k is the *modulus* and ϕ is the upper limit of integration in radians. The value of k is between zero and one and can be real or imaginary. The complete elliptic integral of the first kind is defined when $\phi = \pi/2$. It can be calculated in *Mathematica* with the command **EllipticK[k^2]**.

Elliptic functions were introduced by inverting elliptic integrals. Let u = $F(\phi,k)$. Then Jacobi defines the following basic functions

[4] $$\phi = am(u,k)$$

[5] $$sn(u,k) = \sin(am(u,k)) = \sin(\phi)$$

Other functions can be defined but the **sn** function is all that is required for this analysis.

If u is the complete elliptic integral, then $\sin(\phi) = 1$ in [5]. As with trigonometric functions the inverse of the **sn** function can be defined as

[6] $$Inverse(sn(1,k)) = u = F(1,k) = K(k)$$

The inverse **sn** function is calculated in *Mathematica* with the command **InverseJacobi[1, k^2].**

Ramanujan showed that any positive integer n can be represented by the ratio of complete elliptic integrals;

[7] $$\frac{K(1-k^2)}{K(k^2)} = \sqrt{n}$$

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[1] A. Erdelyi (ed.), **Higher Transcendental Functions**, Volume II, Bateman Manscript Project, McGraw-Hill, (1953)

where k^2 is a real modulus in [3]. The denominator is normally expressed with k' = $\sqrt{1-k^2}$.

From [7] the value of k for a given integer n can be found in *Mathematica* using the **FindRoot** command;

FindRoot[(InverseJacobiSN[1,1 − k^2]/(InverseJacobiSN[1, k^2]))^2 == n, {k, 0.01}, WorkingPrecision → 100]

Since k is associated with an algebraic equation [2] we expect k to be an algebraic integer. Since $|u(\tau)|$ the modulus of the q octic fraction is also an algebraic integer we might expect a connection between these moduli. Examining the fourth power of $|u(\tau)|$ shows this to be true. For odd values and even of n, I find

[8] $$\frac{|u(\tau)|^4}{4} = k * \sqrt{1-k^2}$$

when $|u(\tau)| = |u(\frac{1+\sqrt{-n}}{4})|$, n is an odd integer.

[9] $$\frac{|u(\tau)|^4}{4} = \frac{k}{1-k^2}$$

when $|u(\tau)| = \sqrt{2} * \left(\mathrm{u}(\frac{\sqrt{-n}}{4}) / \left| \mathrm{u}\left(\frac{1+\sqrt{-\frac{n}{4}}}{4}\right) \right|^{\frac{1}{3}} \right)^3$, n is an even integer.

Both [8] and [9] can be rearranged for $|u(\tau)|$ resulting in simple equalies for any integer n;

[10a, b] $$|u(\tau)| = \sqrt{2}\,(k * k')^{1/4} \quad \text{and} \quad |u(\tau)| = \sqrt{2}\,\left(\frac{k}{k'^2}\right)^{1/4}$$

where k' = $\sqrt{1-k^2}$.

Using the Ramanujan octave we can find class invariants G_n and g_n from [10a] and [10b], respectively by calculating $|u(\tau)|^{1/3}$.

[11] $$G_n * |u(\tau)|^{1/3} = G_n * 2^{\frac{1}{6}}\,(k * k')^{\frac{1}{12}} = 2^{\frac{1}{12}} \text{ implies } G_n = 2^{-1/12}\,(k * k')^{-1/12}$$

[12] $$g_n * |u(\tau)|^{1/3} = g_n * 2^{\frac{1}{6}}\left(\frac{k}{k'^2}\right)^{\frac{1}{12}} = 2^{\frac{1}{12}} \text{ implies } g_n = 2^{-1/12}\left(\frac{k}{k'^2}\right)^{-1/12}$$

As in the previous Chapter there is a similar connection of the k modulus to the octahedron. An edge of the octahedron decreases as n increases and can be calculated from the invariant,

[13] $$G_n{}^4 * (edge)^2 = 9^{1/3}$$

[14] $$edge = 3^{1/3} * (2k * k')^{1/6} \text{ and } edge = 3^{1/3} * \left(2\frac{k}{k'^2}\right)^{1/6}$$

with n odd or even respectively.

The volume of the octahedron is V = $|u(\tau)|^2 = 2\,(k * k')^{1/2}$ for odd n and V = $2\,(\frac{k}{k'^2})^{1/2}$ if n is even.

Equation [10b] is used to develop a radical solution to g_n. Let n = 14 and use FindRoot from [7] to find k and k'. From [10b] calculate $(\frac{2k}{k'^2})^{1/4}$ and find the root approximation. *Mathematica* calculates the root of $1 - 8z^2 + 10z^4 - 8z^6 + z^8 = 0$ in radical form and [10b] is

[15] $$|u(\tau)| = 2^{1/4}\sqrt{2+\sqrt{2}-\sqrt{5+4\sqrt{2}}}$$

From [10b] or [12] it is shown that

[16] $$g_n = \frac{1}{(2+\sqrt{2}-\sqrt{5+4\sqrt{2}})^{1/6}} = \sqrt{\frac{1}{2}(1+\sqrt{2}+\sqrt{-1+2\sqrt{2}})}$$

The Elliptic Theta Function

Inverting equations [8] and [9] the modulus k can be expressed as the modulus of the octic q continued fraction.

[17] $$k^2 = \frac{1}{4} * (2 - \sqrt{4 - |u(\tau)|^8})$$

[18] $$k^2 = \frac{8+|u(\tau)|^8 - 4\sqrt{4+|u(\tau)|^8}}{|u(\tau)|^8}$$

respectively for odd and even n.

The square of the modulus k is an important function since it can be expressed by Elliptic modular functions. Of the four types of theta functions only two are needed for this analysis. Theta functions are a series expansion of powers of the nome $q = e^{-\pi\sqrt{n}}$. The theta functions are given in *Mathematica* by **EllipticTheta[a, v, q]** where a = 1, 2, 3 or 4 and v is an argument which for evaluation of k^2 is zero. The two Theta functions required are;

[19] $$\theta_2(0) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2}$$

[20] $$\theta_3(0) = \sum_{n=-\infty}^{\infty} q^{(n)^2}$$

We then have an inverse of the ratio of the complete elliptic integrals [7] for the k modulus.

[21] $$k^2 = \frac{\theta_2(0)^4}{\theta_3(0)^4} = \lambda(\tau)$$

Comparing [21] with [17] and [18] we find that the modulus $|u(\tau)|$ is a function of the ratio of theta functions but with a different nome. Remember that the representation for the nome of $|u(\tau)|$ is $q = e^{2\pi i(1+\sqrt{-n})/4}$ with $\tau = (1+\sqrt{-n})/4$, a complex number whereas, q is real in [19] and [20]. The function [21] is an elliptic modular function and shares the same properties as the eta functions and the Weierstrass elliptic functions. For example, the j-invariant found for various discriminants of elliptic curves can be obtained from $\lambda(\tau)$.

[22] $$j(\tau) = \frac{4}{27} * \frac{(1-\lambda(\tau)+\lambda(\tau)^2)^3}{\lambda(\tau)^2(1-\lambda(\tau))^2}$$

This j-invariant also appears in a previous form for the q octic continued fraction a complex number $u(\tau)$, the octahedral equation,

[23] $$(u(\tau)^{16} + 14 * u(\tau)^8 + 1)^3 - (2^{-4}) * \mathrm{j}(\tau) * (u(\tau)^8 * (u(\tau)^8 - 1)^4) = 0$$

with $u(\tau)$ calculated from $q_1 = e^{2\pi i(\sqrt{-n})}$. If [17] or [18] are substituted into [21] and [22] both odd and even values of n lead to the *same* equation with the modulus of $u(\tau)$ using q for $j(\tau)$.

[24] $$j(\tau) = \frac{(16 - |u(\tau)|^8)^3}{108|u(\tau)|^{16}}$$

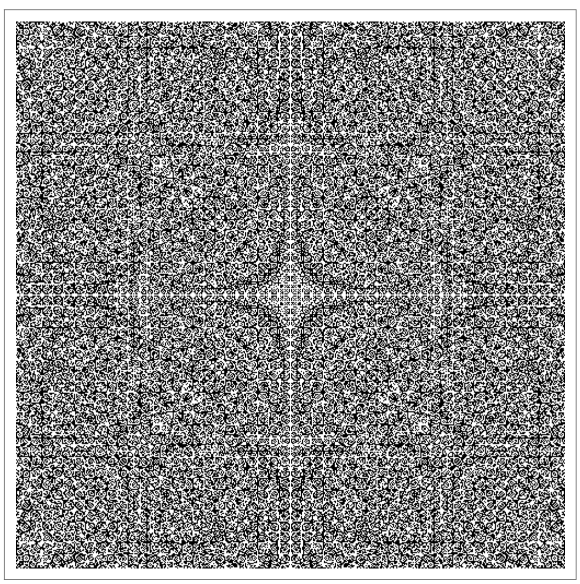
Equations [23] needs to be divided by 1728 = 12^3 in order to equal the value of [24]. The two values of $u(\tau)$ calculated from $u(q_1)$ and $|u(q)|$ are related by the nested radical expression,

[25] $$u(q_1) = \sqrt{\frac{2}{|u(q)|^4} - \frac{\sqrt{2}\sqrt{2-|u(q)|^4}}{|u(q)|^4} - \sqrt{-1 + \frac{8}{|u(q)|^8} + \frac{2}{|u(q)|^4} + \frac{2\sqrt{2}}{\sqrt{2-|u(q)|^4}} - \frac{8\sqrt{2}}{|u(q)|^8\sqrt{2-|u(q)|^4}}}}$$

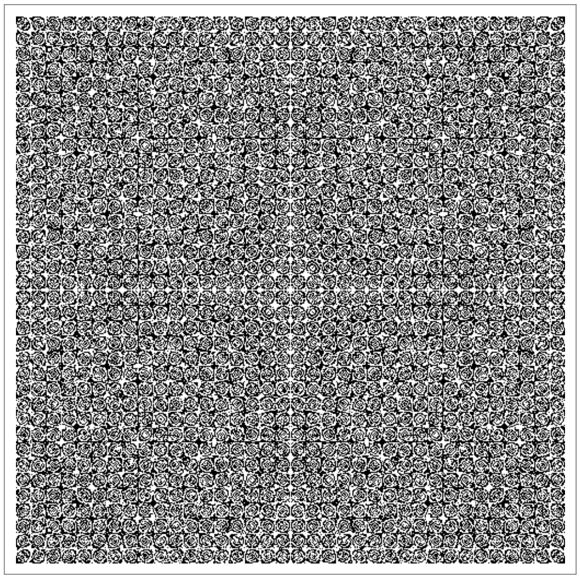
The q octic continued fraction and its modulus is an important modular function taking its form from of the class invariant. It is connected in an interesting manner to the elliptic functions, elliptic integrals and the elliptic Theta function as well as the Dedekind eta function. Of importance, it also shows a connection to the platonic solid, the octahedron and to equal temperament harmonics extending the role played by modular functions into possible new applications.

Since the elliptic functions and integrals are doubly periodic some interesting patterns can be generated in 2 dimensions. Converting the incomplete elliptic integral to digital rounded values mod (2) produces the interesting patterns shown below using k' calculated from n = 23 and n = 14.

N = 23



$N = 14$



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