The q-Octic Transforms

In the previous chapter I showed how the j-invariant could be expressed in two different forms using the q octic continued fraction. The first form appears in the octahedral equation,

[1] $$(u(\tau)^{16} + 14 * u(\tau)^8 + 1)^3 - (2^{-4}) * \mathrm{j}(\tau) * (u(\tau)^8 * (u(\tau)^8 - 1)^4) = 0$$

with $u(\tau)$ calculated as a complex or real number. If $q_1 = e^{2\pi i\tau}$ with $\tau = (\sqrt{-n})$ then $u(\tau)$ is a real number and I define $u(\tau_R)$ as the q octic fraction used to calculate $\mathrm{j}(\tau_R)$ in [1] with the result divided by 1728. The second form of the j-invariant is expressed by the modulus of the q-octic continued fraction. If $q = e^{2\pi i\tau}$ with $\tau = (1+\sqrt{-n})/4$ then $u(\tau)$ is a complex number and the modulus $|u(\tau)|$ is a real number. I define $|u(\tau)| = u(\tau_C)$ as the q octic fraction used to calculate $\mathrm{j}(\tau_C)$ in [2].

[2] $$j(\tau_C) = \frac{(16 - u(\tau_C)^8)^3}{108u(\tau_C)^{16}}$$

Since $j(\tau_C) = \mathrm{j}(\tau_R)$ the two values of $u(\tau)$ calculated from $u(\tau_R)$ and $u(\tau_C)$ are equated by the nested radical expression,

[3] $$u(\tau_R) = \sqrt{\frac{\sqrt{2}\sqrt{2-u(\tau_C)^{\ 4}}}{u(\tau_C)^{\ 4}} - \frac{2}{u(\tau_C)^{\ 4}} + \sqrt{-1 + \frac{8}{u(\tau_C)^{\ 8}} + \frac{2}{u(\tau_C)^{\ 4}} + \frac{2\sqrt{2}}{\sqrt{2-u(\tau_C)^{\ 4}}} - \frac{8\sqrt{2}}{u(\tau_C)^{\ 8}\sqrt{2-u(\tau_C)^{\ 4}}}}}$$

The inverse of [3] is a more formidable expression,

[4] $$u(\tau_C) = \left(-\frac{128(3u(\tau_R)^8+10u(\tau_R)^{16}+3u(\tau_R)^{24})}{(-1+u(\tau_R)^8)^4} + 64\sqrt{\frac{u(\tau_R)^8+30u(\tau_R)^{16}+255u(\tau_R)^{24}+452u(\tau_R)^{32}+255u(\tau_R)^{40}+30u(\tau_R)^{48}+u(\tau_R)^{56}}{(-1+u(\tau_R)^8)^8}}\right)^{1/8}$$

These equations in functional form are

[5a, b] $u(\tau_R)$ = R[$u(\tau_C)$] and $u(\tau_C)$ = C[$u(\tau_R)$]

Equations [5a] and [5b] are the R and C q-octic transforms. These radical transforms are solvable since the j-invariant is a real number and equal in both equations [1] and [2].

It can be demonstrated that the R transform is real if $2^{\ 1/4} > u(\tau_C) > 0$. For $u(\tau_C) > \ 2^{\ 1/4}$ the R transform is a complex number. The condition on the C transform is that $u(\tau_C) \neq 1$. Also C has the properties ,

[6] $$[C[u(\tau_R)] = \mathrm{C}[1/u(\tau_R)]$$

[7] $$[C[R[u(\tau_C)]] = u(\tau_C)$$

[8] $$[R[C[u(\tau_R)]] = u(\tau_R)$$

Both transforms are symmetric about the origin.

These transforms are expressed in radical form. If the q-octic $u(\tau_R)$ is an integer or a fraction of two integers or a square root of two integers, then the C transform is an algebraic number of a minimal polynomial of order four. Some examples are:

$\mathrm{C}[5] = \mathrm{C}[1/5] = 2\sqrt{\frac{5}{313}}78^{1/4}$ Minimal polynomial $-31200 + 97969x^4 = 0$

$\mathrm{C}[59/29] = \mathrm{C}[29/59] = 2\sqrt{\frac{3422}{6412321}}356565^{1/4}$ Minimal polynomial $-876723930720 + 896053453201x^4 = 0$

$\mathrm{C}\left[\sqrt{\frac{907}{205}}\right] = \mathrm{C}[\sqrt{\frac{205}{907}}] = \frac{23^{3/4}671969090^{1/4}}{\sqrt{432337}}$ Minimal polynomial $-290290646880 + 186915281569x^4 = 0$

If the ratio of two integers is a higher root or power, then the solution can be obtained from a transformed fourth order polynomial;

$\mathrm{C}[\sqrt[5]{31}] = \mathrm{C}\left[\frac{1}{\sqrt[5]{31}}\right] = Root\ of\ \ -7270422282240 + 18913710080x^4 + 1704005222400x^8 + 1477633600x^{12} - 44375136000x^{16} + 213223221121x^{20} = 0$

The R transform is a solution of an eight-order polynomial if the q-octic $u(\tau_C)$ is an integer or a fraction of two integers or a square root of two integers.

$\mathrm{R}[1/5] = \sqrt{-1250 + 25\sqrt{2498} + \sqrt{3126249 - 62550\sqrt{2498}}}$ Minimal polynomial $1 - 5000x^2 + 2x^4 + 5000x^6 + x^8 = 0$

$R[29/59]$ Minimal polynomial $707281 - 96938888x^2 + 1414562x^4 + 96938888x^6 + 707281x^8 = 0$

$\mathrm{R}[\sqrt{\frac{205}{907}}]$ Minimal polynomial $42025 - 6581192x^2 + 84050x^4 + 6581192x^6 + 42025x^8 = 0$

An interesting series is obtained from the R transform for the inverse of a number or fraction N. Evaluate $R[1/(N * 2^m)]$ for any integer m (zero, positive or negative). The value of the transform is then a solution to the eight-order equation $1 - Ax^2 + 2x^4 + Ax^6 + x^8 = 0$. The coefficient A is given by

[9] $$A = 2^{(3+4m)} * N^4$$

A similar series is obtained from the C transform for an integer K. Evaluate C[[1/(K*K^m)] for any integer m (zero or positive). The value of the transform is a solution to the fourth order equation, -D+Bx^4 = 0. The coefficients D and B are given by,

[10] $$B = (((K * K^m)^4 + 1)/2)^2$$

[11] $$D = D_0 * K^{2m}(1 + \sum_1^m K^{4m})$$

where D_0 is the D coefficient when m =0 calculated from the minimal polynomial of C[[1/K]. (see example with C[1/5] above)

Equations [1] or [2] could be used to find the radical from of the j-invariant. For example j[$\sqrt{-14}$] is found from $u(\tau_C)$, equation [31-15] from the previous chapter.

[31-15] $$|u(\tau)| = u(\tau_C) = 2^{1/4}\sqrt{2 + \sqrt{2} - \sqrt{5 + 4\sqrt{2}}}$$

Substituting into [2] and multiplying by 1728 gives

[12] $$j(\tau_C) = \frac{(16-4(2+\sqrt{2}-\sqrt{5+4\sqrt{2}})^4)^3}{(2+\sqrt{2}-\sqrt{5+4\sqrt{2}})^8} = 1.620803339..\times 10^{10}$$

The same value is obtained using the transform j[$R[u(\tau_C)]$] however the radical expression is not as compact as in [12].

The calculation of the j-invariant is not as simple as substituting q-octic values into [1] or [2]. Based on a theorem in Cox[1] the actual value calculated for the q octic fraction may be complex. If the discriminant of a quadratic field is equal to -4m = 0 mod 4, then the values for $u(\tau_C)$ and $u(\tau_R)$ are real. However, if the discriminant = -m = 1 mod 4 then the values for $u(\tau_C)$ and $u(\tau_R)$ are imaginary or complex. Fortunately, the q octic transforms are true for both real and complex arguments. As two examples consider $\tau = \sqrt{-2})$ and $\tau = (3+\sqrt{-7})/2$. The first giving a real $u(\tau_R)$ since -4*2 = 0 mod 4 and the second an imaginary $u(\tau_R)$ since -7 = 1 mod 4. Calculating the first, I obtain,

[13] $$u\left(\tau_R(\sqrt{-2})\right) = 0.4656665496226619848\,...$$

$$\text{and } \mathrm{C}[u\left(\tau_R(\sqrt{-2})\right)] = 1.10816100896431157997..$$

Then from [1] and [2] I calculate (multiplying by 1728),

[14] $$j\left(u\left(\tau_R(\sqrt{-2})\right)\right) = j\left(\mathrm{C}[u\left(\tau_R(\sqrt{-2})\right)]\right) = 8000 = 20^3$$

From the second example complex values are found,

[15] $$u\left(\tau_R((1+\sqrt{-7})/2)\right) = 0.462393768686590.. + 0.191529770146793..i$$

$$\text{and } \mathrm{C}[u\left(\tau_R((1+\sqrt{-7})/2)\right)] = 1.1759685821064740.. + 0.17695043450..i$$

Then again from [1] and [2] I calculate the real j-invariant (multiplying by 1728),

[16] $$j\left(u\left(\tau_R((1+\sqrt{-7})/2)\right)\right) = j\left(\mathrm{C}[u\left(\tau_R((1+\sqrt{-7})/2)\right)]\right) = -3375 = -15^3$$

The values in [14] and [16] agree with literature values and with reference (1). In this case where the q-octic is imaginary, it is important to find $u(\tau_R)$ first and then transform using $\mathrm{C}[u(\tau_R)]$ to obtain the corresponding value of $u(\tau_C)$.

The q-octic transform is also modular-like and multi-valued. If the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ with determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$ then

[17] $$\mathrm{C}[u_R[\tfrac{a\tau+b}{c\tau+d}]] = \mathrm{C}[u_R[\tau]]$$

where $u_R[\tau]$ is real but $u_R[\frac{a\tau+b}{c\tau+d}]$ can be real, complex or imaginary. Since multiple values of $u_R[\frac{a\tau+b}{c\tau+d}]$ represent a single value of $u_C[\tau]$ the q octic transform is applied to many different quadratic fields τ. Any quadratic field can then be represented by $u_C[\tau]$ and a corresponding class invariant. For example, the quadratic fields $[\frac{\tau+2}{2\tau+5}]$, $[\frac{3\tau+1}{2\tau+1}]$, $[\frac{7\tau+1}{13\tau+2}]$, $[\frac{2\tau+1}{7\tau+4}]$ where $\tau = (\sqrt{-23})$ are all represented by $u_C[(1+\sqrt{-23})/4]$ and the class invariant G_{23} and minimal polynomial $x^3-x-1 = 0$.

For example, consider the field $[\frac{3\tau-5}{-\tau+2}]$, with $\tau = (\sqrt{-31})$ then I find $u_R[\frac{3\tau-5}{-\tau+2}]$ = 1.00031829960438614679265464.. and $\mathrm{C}[u_R[\frac{3\tau-5}{-\tau+2}]]$ = 0.2246281641076307375848.. This q-octic is exactly the value for $u_C[(1+\sqrt{-31})/4]$ and it can easily be shown using the Ramanujan octave that it leads to the class invariant and real solution for $x^3- x^2-1 = 0$. If the quadratic field leads to a complex

[1] David Cox, Primes of the form $x^2 + ny^2$, John Wiley and Sons (2013), pp 227.

solution for $u_R\,[\frac{a\tau+b}{c\tau+d}]$, it is important that the magnitude of this solution does not equal 1 since C[1] is indeterminant. Also, $u_R\,[\frac{-1}{\tau}]$ is the complementary real value where $u_R\,[\frac{-1}{\tau}] \neq u_R\,[\tau]$ but $C[u_R\,[\frac{-1}{\tau}]] = C[u_R\,[\tau]]$.

This equation for complementary real illustrates an interesting property of the q transformation. For two numbers x and y

[18] $$C\left[\sqrt{\frac{x}{y}}\right] = C\left[\sqrt{\frac{y-x}{x+y}}\right]$$

[19] $$C\left[\frac{x}{y}\right] = C\left[\sqrt{\frac{y^2-x^2}{x^2+y^2}}\right]$$

In [18] the q transform takes the fraction formed by two numbers and forms the fraction of the difference divided by the sum of the two numbers. As an example, if x= 7 and y = 13 then,

$$C\left[\sqrt{\frac{7}{13}}\right] = \frac{21365^{1/4}}{\sqrt{109}} \quad \text{with } R\left[\frac{21365^{1/4}}{\sqrt{109}}\right] = \sqrt{\frac{13-7}{7+13}} = \sqrt{\frac{3}{10}} \text{ then } C\left[\sqrt{\frac{3}{10}}\right] = \frac{21365^{1/4}}{\sqrt{109}}$$

And in [19]; $C\left[\frac{7}{13}\right] = 2\sqrt{\frac{91}{15481}}\,3270^{1/4}$ then $C\left[\sqrt{\frac{13^2-7^2}{7^2+13^2}}\right] = C\left[2\sqrt{\frac{15}{109}}\right] = 2\sqrt{\frac{91}{15481}}\,3270^{1/4}$

This property indicates that the j-invariant can take the same value from two different q octic continued fractions or two different quadratic fields! In the example, finding j calculated from the octahedral form [1] above;

[20] $$j\left[\sqrt{\frac{3}{10}}\right] = j\left[\sqrt{\frac{7}{13}}\right] = 1.63023..$$

The q transforms preserve the complete elliptic integral of the first kind described in the previous chapter, $K[k^2]$. Based on [6] and [19] the original value of elliptic integral and the complementary values are shown in the equation below for the value of k giving the integer m as calculated from the ratio of complete elliptic integrals.

[21[$$\frac{K(1-k^2)}{K(k^2)} = \sqrt{m} = \frac{C[R[K(1-k^2)]]}{C[R[K(k^2)]]} = \frac{R[C[K(k^2)]]}{R[C[K(1-k^2)]]}$$

One other observation is evaluation of complex values of $C[u_R\,[\frac{a\tau+b}{c\tau+d}]]$. As above these values can be used for finding the j-invariant, however it is questionable how to apply the complex value to find G_m and the minimal polynomial. Fortunately, the average of the modulus is used for these calculations. For example, the class 1 discriminant and quadratic field are used to find the j-invariant. I find that $C[u_R\,[\frac{\tau+3}{2}]]$ where $\tau = (\sqrt{-67})$ is the complex number $u_c = 0.39319750180623407991 + 0.078208649742213662882 6i$. Plugging into [2] and multiplying by 1728 gives the accepted number 5280^3 as the j invariant Reference (1) pg. 237. If the modulus of this number is calculated and divided by 2 then the correct real value of u_c is found and when used with the Ramanujan octave finds the correct solution to the minimal polynomial $-2 - 2z - 2z^2 + z^3 = 0$!

Other q-Octic Transforms

The list of transforms is complete by adding the transform between the k square modulus and the value of $u(\tau_R)$ and $u(\tau_C)$. From the last Chapter, k or k2 could be calculated from the ratio of two Theta functions;

[22] $$k^2 = \frac{\theta_2(0)^4}{\theta_3(0)^4} = \lambda(\tau)$$

With the nome for τ as $q(\tau) = e^{-\pi\sqrt{n}}$.

It can be shown by comparing the octahedral and k modular forms of the j invariant that the q- octic continued fraction is

[23] $$\mathrm{RL}[k^2] = (1 + \frac{8}{\lambda^2} + \frac{4\sqrt{-(-2+\lambda)^2(-1+\lambda)}}{\lambda^2} - \frac{8}{\lambda})^{-1/8}$$

With $u(\tau_R) = \mathrm{RL}[k^2]$

This inverse transform is

[24] $$k^2 = \mathrm{L2}[u(\tau_R)] = \frac{4u(\tau_R)^4}{(1+u(\tau_R)^4)^2}$$

For n an odd and positive integer equations [23], [24], [3] and [4] are compatible with $\mathrm{RL}[k^2] = \mathrm{R}[u(\tau_C)] = u(\tau_R)$.

For even values of n $\mathrm{RL}[k^2] \neq \mathrm{R}[u(\tau_C)]$ since the calculation of $u(\tau_C)$ requires the ratio of $\frac{k}{1-k^2}$. We know that for n even,

[25] $$u(\tau_C) = \sqrt{2} * (k/(1-k^2))^{1/4}$$

Where k^2 is found from [22]. Then $\mathrm{R}[u(\tau_C)] = u(\tau_R)$. The values of the j-invariant are slightly different if $\mathrm{RL}[k^2]$ or $\mathrm{R}[u(\tau_C)]$ are used in $j(u(\tau_R))$. The value of $\mathrm{RL}[k^2]$ is compatible with the value obtained using the **KleinInvariantJ** command in *Mathematica* to calculate the Klein Invariant modular elliptic function. Although this value also agrees with values obtain by Cox (pg. 246) the value obtained using $\mathrm{R}[u(\tau_C)]$ is better suited when the q-octic continued fraction and the Ramanujan octave are used to calculate the class invariant value g_n for even n and the associated minimal polynomial for the class field.

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