Geometry of the Perrin and Padovan Sequences II

Building a Perrin Sequence

This chalkboard further examines the geometry of the Perrin sequence which was previously discussed in Chapter 19. Another recent publication[1] discusses the geometric interpretation of the Perrin and Padovan numbers using multi-variate polynomials. I will use previous theorems 19.1 and 19.2 to prove the result in reference [1]. This result is then extended to other sequences related to the Perrin sequence derived from the equation $x^3 - x - 1 = 0$. A second morphic number derived from the equation $x^3 - x^2 - 1 = 0$ will also be demonstrated to find the so-called Narayana's cow sequence[2] and other related sequences. The construction of the plastic number leads to the following previously derived theorems:

19.1 Theorem: ***The Perrin sequence can be generated from a unit measure and powers of the plastic number, ψ. Given a unit measure a paper folding technique can be used to construct ψ and powers of ψ. All Perrin numbers can be generated from the unit measure (1) and the three constructible numbers ψ, $1/\psi$, and ψ^{-5}.***

Corollary*: Let ψ_1 and ψ_2 be the complex and complex conjugate solution to the equation $x^3 - x - 1 = 0$. All powers $\psi_1^n + \psi_2^n$ can be generated from the real numbers 2, $-\psi$ and ψ^{-5}.*

The Perrin sequence as shown in previous chapters is generated from powers of the three solutions to the cubic equation.

[19-4] $\quad$ **P(n)** $= \psi^n + \psi_1^n + \psi_2^n$

where n = 0, 1, 2, …. n.

Let the powers of the real and complex solutions be constructed as follows:

$\psi_1^0 + \psi_2^0 = 2$ and $\psi^0 = 1$

$\psi_1^1 + \psi_2^1 = -\psi$ and $\psi^1 = \psi$

$\psi_1^2 + \psi_2^2 = \psi^{-5}$ and $\psi^2 = 1 + 1/\psi$

19.2 Theorem: Given the three generators for n = 0, 1, and 2 all powers are obtained from the recurrence relation $\psi_1^n + \psi_2^n = \psi_1^{n-2} + \psi_2^{n-2} + \psi_1^{n-3} + \psi_2^{n-3}$ and $\psi^n = \psi^{n-2} + \psi^{n-3}$. The Perrin sequence P(n) is completely generated with positive and negative powers of ψ^n.

As in a previous example using Theorems 1 and 2, generate and show that the Perrin number P(8) = 10.

From Theorem 1 we can show that[3]:

$\psi^8 = 1 + \psi + 1 + 1/\psi + 1 + \psi + \psi + 1 + 1/\psi = 4*\mathbf{1} + 3*\boldsymbol{\psi} + 2*\mathbf{1/\psi} = 9.48390920$

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[1] M. Artioli and G. Dattoli, Geometric Interpretation of Perrin and Padovan Numbers., website: http://demonstrations.wolfram.com/GeometricInterpretationOfPerrinAndPadovanNumbers/ Sept. 2016.

[2] On-line Encyclopedia of Integer Sequences,OEIS, A000930 ' Narayana's cow sequence', https://oeis.org/

[3] Also see Richard Turk in OEIS A001608 for P(5)

From Theorem 2

$\psi_1^8 + \psi_2^8 = \psi^{-5} + 2 - \psi + 2 - \psi - \psi + \psi^{-5} = 2*\boldsymbol{\psi}^{-5} + 2*\mathbf{2} - 3*\boldsymbol{\psi} = 0.51609080$

$\psi^8 + \psi_1^8 + \psi_2^8 = 9.48390920 + 0.51609080 = 10.00000000$

If one examines higher powers, we find that the coefficients found before ***ψ, 1/ψ, and ψ^-5^*** are following a Padovan sequence[4]. In my analysis I remove the first three terms of the Padovan sequence. Let Pv(n) be the nth Padovan number of the sequence 1,0,1,1,1,2,2,3,4,5,7,9,12,16, …. then in the example above with n = 8,

$(2*\mathbf{2} - 3*\psi + 2*\psi^{-5}) + (4*\mathbf{1} + 3*\psi + 2*\mathbf{1/\psi}) = (2*\boldsymbol{Pv}[\boldsymbol{n}-\mathbf{3}] - \boldsymbol{Pv}[\boldsymbol{n}-\mathbf{1}]\boldsymbol{\psi} + \boldsymbol{Pv}[\boldsymbol{n}-\mathbf{2}]\boldsymbol{\psi}^{-5}) + (\boldsymbol{Pv}[\boldsymbol{n}] + \boldsymbol{Pv}[\boldsymbol{n}-\mathbf{1}]\boldsymbol{\psi} + \boldsymbol{Pv}[\boldsymbol{n}-\mathbf{2}]/\boldsymbol{\psi}) = \mathbf{P}(8) = 10$

19.3 Theorem*: The Padovan sequence is generated from the Perrin sequence after subtraction of the integer generators of powers of the real and complex solutions. The remaining metrics $1/\psi$ and ψ^{-5} sum to a unit measure. Conversely, the Padovan sequence generates the Perrin sequence.*

The authors in reference [1] derive a multivariate Legendre function $P_n[x, y, z]$ where the nth term is a function of the values of x, y, and z. In their derivation of equation [1] below they set z = 0 and let $P_n[x, y, 0] = P[n,x,y]$

$$[1]\; P[\mathrm{n,x,y}] = \sum_{r=0}^{\lfloor n/3 \rfloor} \frac{(-x)^r(-y)^{\lfloor (n-3r)/2 \rfloor}}{\lfloor (n-3r)/2 \rfloor ! r!} \mathrm{Gamma}[r + \lfloor (n-3r)/2 \rfloor + 1] * \mathrm{Abs}[\mathrm{Cos}[(n-3r)*\pi/2]]$$

With x = -1 and y = -1 and calculating P[n, -1, -1], this function generates the Padovan sequence. In the example above, we find that the $\boldsymbol{Pv}[\boldsymbol{n}-\mathbf{1}]\boldsymbol{\psi}$ terms always cancel and the coefficients $\boldsymbol{Pv}[\boldsymbol{n}-\mathbf{2}]$ are always equal for the metrics $\mathbf{1/\psi}$ and ψ^{-5}. We remember from the construction of these numbers that $\mathbf{1/\psi} + \psi^{-5} = 1$. This leaves the following terms and equality for any Perrin number P(n):

[2] P(n) = $\mathbf{2} * \boldsymbol{P}[\boldsymbol{n}-\mathbf{3},-\mathbf{1},-\mathbf{1}] + \boldsymbol{P}[\boldsymbol{n},-\mathbf{1},-\mathbf{1}] + \boldsymbol{P}[\boldsymbol{n}-\mathbf{2},-\mathbf{1},-\mathbf{1}] = \mathbf{3} * \boldsymbol{P}[\boldsymbol{n},-\mathbf{1},-\mathbf{1}] - \boldsymbol{P}[\boldsymbol{n}-\mathbf{2},-\mathbf{1},-\mathbf{1}]$

In *Mathematica* we can show that a table of coefficients P[n, -1, -1] for integers n follow the series expansion of the generating function $1/(1+zt+yt^2+xt^3)$ as shown in [3] where z = 0, y = -1 and x = -1,

[3] **Series[1/(1 + z* t + y*t^2 + x*t^3), {t, 0, 18}] =**

$$1 + t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + 4t^8 + 5t^9 + 7t^{10} + 9t^{11} + 12t^{12} + 16t^{13} + 21t^{14} + 28t^{15} + 37t^{16} + 49t^{17} + 65t^{18} + O[t]^{19}$$

This series is the Padovan sequence. Another code for generating these coefficients without the variable t^n in *Marthematica* is [4].

[4] $\mathbf{LinearRecurrence}[\{\mathbf{0},\mathbf{1},\mathbf{1}\},\{\mathbf{1},\mathbf{0},\mathbf{1}\},\mathbf{18}]$ =

{1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49}

Reading off this list we see as an example that P(16) = 3*P[16,-1,-1] – P[14, -1,-1] = 90. Looking at the multivariate Legendre polynomial from [1] and [2], we find that for both terms n = 16 and n = 14,

[5] P(16) = 3*($15x^4y^2 - 21x^2y^5 + y^8$) – $(-5x^4y + 15x^2y^4 - y^7)$

[4] See OEIS A000931.

which is 90 when y = -1 and x = -1.

Sequences for the general equation $x^3 - R2*x - R1 = 0$

The above analysis for the Perrin sequence and associated cubic polynomial $x^3 - x - 1 = 0$ is a general case for generating sequences from the general polynomial $x^3 - R2*x - R1 = 0$ where R1 and R2 can be positive or negative fractions or integers. Let the function P[n, x, y] be generalized as P[n, -R1, -R2]. Then this function is the generating sequence of $x^3 - R2*x - R1 = 0$ just as P[n, -1, -1] is the generating Padovan sequence for $x^3 - x - 1 = 0$!.

As an example, let x = -R1 = -3 and y =-R2 = -4 and find the 16th term of the sequence for the cubic monic polynomial $x^3 - 4*x - 3 = 0.$ Using the similar two-dimensional Legendre polynomials as in [5] above, it can be shown that the second term is multiplied by R2 and finds the solution.

[6] $P_{(3,4)}(16) = 3*(\ 15x^4y^2 - 21x^2y^5 + y^8) - R2*(-5x^4y + 15x^2y^4 - y^7)$

which is 625280 when y = -4 and x = -3 and R2 = 4. In general $P_{(R1,R2)}(n)$ is the n^{th} term of the sequence associated with the cubic equation $x^3 - R2*x - R1 = 0$. This substantiates the following theorem:

Theorem 19.4 -The nth term of the sequence associated with the cubic equation $x^3 - R2*x - R1 = 0$ is expressed by P[n, -R1, -R2] and calculated from the equation $P_{(R1,R2)}(n)$ = 3P[n, -R1, -R2] - R2* P[n-2, -R1, -R2].

In the next section I discuss the Narayana's Cow sequence. This sequence is said to have originated from a 14th century Indian mathematician, Narayana. A cow produces a calf every year and after the fourth year each calf produces a calf beginning each year. How many total cows and calves are there after 10, after 20 years? The Narayana sequence is found to be reminiscent of the Padovan sequence because it generates a new sequence like the Perrin sequence.

Sequences for the general equation $x^3 - M2* x^2 - M1 = 0$

In *Mathematica* the coefficients of Narayana can be found from the following,

[7] **LinearRecurrence**[{**1**, **0**, **1**}, {**1**, **1**, **1**}, **20**] =

{1,1,1,2,3,4,6,9,13,19,28,41,60,88,129,189,277,406,595,872}

where we find the answer of 19 and 872 cows and calves to his question, respectively. The new morphic number derived from the cubic equation $x^3 - x^2 - 1 = 0$ provides new generators for finding the nth term of the new sequence. The OEIS does not have a name for this sequence, however it is generated from the three numbers solving the equation, $x^3 - x^2 - 1 = 0$. Following the analysis for the Perrin sequence this new sequence N(n) will be generated from powers of the three solutions to the cubic equation.

[8] $$\mathbf{N(n)} = \phi^n + \phi_1^n + \phi_2^n$$

where n = 0, 1, 2, …. n.

Let the powers of the real and complex solutions be constructed as follows:

$\phi_1^0 + \phi_2^0 = 2$ and $\phi^0 = 1$

$\phi_1^1 + \phi_2^1 = 1-\phi$ and $\phi^1 = \phi$

$\phi_1^2 + \phi_2^2 = \phi^{-3} - \phi$ and $\phi^2 = \phi + 1/\phi$

where ϕ = 1.4655712318…is a real algebraic number[5].

19.5 Theorem: Given the three generators for n = 0, 1, and 2 all powers are obtained from the recurrence relation $\phi_1^n + \phi_2^n = \phi_1^{n-1} + \phi_2^{n-1} + \phi_1^{n-3} + \phi_2^{n-3}$ and $\phi^n = \phi^{n-1} + \phi^{n-3}$. The sequence N(n) is completely generated with positive and negative powers of ϕ^n.

The result for the nth term N(n)is shown to be generated from Narayana's sequence which I label as **a[n] (see below);**

[9] $N(n) = ((a[n+1] - (a[n-1] - a[n-2])) - a[n]\phi + a[n-1]\phi^{-3}) + (a[n-2] + a[n]\phi + a[n-1]/\phi)$

Combining terms and using the fact that ***$1/\phi + \phi^{-3}$*** = 1, results in the equation

[10] $N(n) = a[n+1] + 2a[n-2] = 3a[n+1] - 2a[n]$

It can be shown that as in the case of the Perrin equation a general result is obtained shown in the next theorem;

Theorem 19.6 -The nth term of the sequence associated with the cubic equation $x^3 - M2^* x^2 - M1 = 0$ is expressed by a[n, -M1, -M2] and calculated from the equation $N_{(M1,M2)}(n)$ = 3a[n+1, -M1, -M2] – 2M2* a[n, -M1, -M2].

The Narayana sequence in [9[and [10] is **a[n]** = **a[n, -1, -1].** However, we need to return to the Series expression in [3] where now the values are z = -1, y = 0 and x = -1,

[11] **Series[1/(1 + z* t + y*t^2 + x*t^3), {t, 0, 20}] =**

$$1 + t + t^2 + 2t^3 + 3t^4 + 4t^5 + 6t^6 + 9t^7 + 13t^8 + 19t^9 + 28t^{10} + 41t^{11} + 60t^{12} + 88t^{13} + 129t^{14} + 189t^{15} + 277t^{16} + 406t^{17} + 595t^{18} + 872t^{19} + 1278t^{20} + O[t]^{21}$$

The coefficients of the first 3 terms generate this sequence and are used to define a[n] with M1 = M2 = 1

[12] **a[n]** = **a[n, -1, -1]** = $\mathbf{LinearRecurrence}[\{\boldsymbol{M2}, \mathbf{0}, \boldsymbol{M1}\}, \{\mathbf{1}, \mathbf{1}, \mathbf{1}\}, \mathbf{20}]$

As an example, if M1 and M2 are not unity let z= M1=5 and x = M2 =4 then

[13] **Series[1/(1 + z* t + y*t^2 + x*t^3), {t, 0, 10}] =**

$$1 + 4t + 16t^2 + 69t^3 + 296t^4 + 1264t^5 + 5401t^6 + 23084t^7 + 98656t^8 + 421629t^9 + 1801936t^{10} + O[t]^{11}$$

[14] **a[n, -4, -5]** = $\mathbf{LinearRecurrence}[\{\boldsymbol{M2}, \mathbf{0}, \boldsymbol{M1}\}, \{\mathbf{1}, \mathbf{4}, \mathbf{16}\}, \mathbf{20}]$

The result shows that the 8th term of the sequence generated from $x^3 - 4{*}x^2 - 5 = 0$ is

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[5] See OEIS A092526.

$N_{(5,4)}(8) = 3a[9, -5, -4] – 2M2* a[8, -5, -4] = 3 * 98656 - 8 * 23084 = 111296$

Since the Legendre polynomial is derived with z = 0 the function P[n,x,y] cannot be used to calculate a[n]. When z is non-zero the expression given in reference [1] calculates a complex infinity. If the correct expression becomes available from the authors of reference [1] one can use P[n,x,z] with y = 0 as a substitute for a[n,M1,M2].

There is another equation providing values of a[n, M1, 1] found in the OEIS page for Narayana's Cow sequence[6].

[15] $a[n+1,M1,1] = b[n,M1,1] = \text{HypergeometricPFQ}[\{(1-n)/3, (2-n)/3, -n/3\}, \{(1-n)/2, -n/2\}, -27 * (M1)/4]$

where the hypergeometricPFQ function is the general hypergeometric function calculated in *Mathematica*.

$$\text{HypergeometricPFQ}[\{a1, a2, a3\}, \{b1, b2\}, z] = 1 + \frac{a1a2a3z}{b1b2} + \frac{a1(1+a1)a2(1+a2)a3(1+a3)z^2}{2b1(1+b1)b2(1+b2)} + O[z]^3$$

Note that there is a shift in the values between the series and the hypergeometric function such that a[n+1, M1,1] = b[n, M1,1]. Currently there is no formula for calculating b[n, M1,M2] with M2 $\neq$1.

This chalkboard can be useful for future study of integer sequences and as a guide with the use of *Mathematica* to understand and verify these equations. This paper could also be a guide for the following two unsolved problems, finding b[n, M1,M2] with M2 $\neq$1 and the two-dimensional Legendre polynomial P[n,x,z] with y = 0. Also, with a full *Lacunary* Legendre polynomial P[n,x,y,z] the n^{th} term of sequences for cubic polynomials $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$ may be expressed with P[n, $\mathbf{-c_2}$, $\mathbf{-c_1}$,- $\mathbf{c_0}$].

Postscript[7]

[For those who would like to work the above problem this paragraph provides hints as well as a partial answer. Read with care if you do not like spoilers! Again, we are seeking a solution to the general problem for $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$. The answer will require a general solution to the series P[n, $\mathbf{-c_2}$, $\mathbf{-c_1}$,-$\mathbf{c_0}$]. I do not have a formula as in equation [1] above but I do recommend looking at the Tribonacci numbers (OEIS A000073 and A081172]. Steps to the solution:

1. Construct the real and complex solutions with the root to $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$. Use integers for c_0, c_1 and c_2 to check your answer. Hint [Use a combination of real and complex parts such as (2- x^2) + (2+ x + 1/x).] Use equation [8] to check the sequence pattern.
2. The sequence of coefficients with increasing powers follow two different Tribonacci series.
3. Use *Mathematica* to obtain the two sequences $P7 = \text{LinearRecurrence}[\{c2, c1, c0\}, \{1, c2, 0\}, n]$ and $P8 = \text{LinearRecurrence}[\{c2, c1, c0\}, \{0, 1, c2\}, n]$.
4. Show that a general solution to the numbers in the sequence is;

[16] $2 * P7[[n+2]] + P7[[n+1]] - (2c0 - 1) * P8[[n]] + P8[[n]] * ((c2-1)^2 - 2(c2-1) * (c1-1)) - 2c0 * (c2-1) * P8[[n-1]]$

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[6] Attributed to Jean-François Alcover in OEIS A000930.
[7] In memory of my Aunt Florence

Using *Mathematica* we can express P7 and P8 in terms of a polynomial in z, y, and x. For example one can calculate the 5th term of any cubic $x^3 - c_2 x^2 - c_1 x - c_0 = 0$ using the expanded form of P7 and P8, e.g P7[z, y, x] = LinearRecurrence[{z, y, x},{1, z, 0},5], and plug into equation [16];

[17] $-2c0(-1 + c2)(y + z^2) + yz(y + z^2) + x(y + 2z^2) - (2c0 + 2c1(-1 + c2) - c2^2)(x + 2yz + z^3) + 2(x^2 + yz^2(2y + z^2) + 2x(2yz + z^3))$

This polynomial equation is used to find the value of ***any*** 5th term by letting z = c2 = M2, y = c1 = R2 and x = c0 = R.

It still remains to find a general function for P7[[n]] = $P_{(1,c2,0)}[n, -c_2, -c_1, - c_0]$ and P8[[n]] = $P_{(0,1,c2)}[[n, -c_2, -c_1, - c_0]$, potentially from the three dimensional *lacunary* Legendre Polynomials!]

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