Calculus of Integer Sequences

Building Integer Sequence Polynomials

The integer sequence of a general cubic monic polynomial as shown in a previous chapter is generated from powers of the three roots to the cubic equation $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$.

[1] **S(n)** = $\psi^n + \psi_1^n + \psi_2^n$

where n = 0, 1, 2, …. n.

Based on the 'geometry' of these sequences **S(n)** can be built from a single root. It is best to choose the real root or any real root if all the roots are real. Following the method used to build the Perrin sequence and the Narayana cow's sequence, I show that the general sequence for [1] with unit coefficients is constructed from;

$\psi_1^0 + \psi_2^0 = 2$ ***and*** $\psi^0 = 1$

$\psi_1^1 + \psi_2^1 = 1 - \psi$ ***and*** $\psi^1 = \psi$

$\psi_1^2 + \psi_2^2 = 2 - \psi^2$ ***and*** $\psi^2 = 2 + \psi + 1/\psi$

There are two sequences of coefficients which appear at higher powers of n. These are the coefficients found in the Tribonacci series (OEIS A000073 and A081172]. It is not too difficult to find that for any values of $\mathbf{c_2, c_1,}$ **and** $\mathbf{c_0}$ the two series are,

[2] **P7[c_2, c_1, c_0] = P7[z, y, x]** = $\mathbf{LinearRecurrence}[\{z, y, x\}, \{0, 1, z\}, n]$

[3] **P8[c_2, c_1, c_0] = P8[z, y, x]** = $\mathbf{LinearRecurrence}[\{z, y, x\}, \{1, z, 0\}, n]$

The coefficients c_2, c_1, and c_0 can be equated to the variables z, y and x respectively. For any value of the coefficients c_2, c_1, and c_0 the general multivariate polynomial for a given power n is

[4] $2 * \mathrm{P7}[[n+2]] + \mathrm{P7}[[n+1]] + ((z-1)^2 - 2(z-1)*(y-1) - (2x-1)) * \mathrm{P8}[[n]] - 2x * (z-1) * \mathrm{P8}[[n-1]]$

As shown in the postscript of the last chapter for each n there is a polynomial representing the nth term of <u>any</u> sequence with the cubic polynomial coefficients z = c_2, y = c_1, and x = c_0. For example, for n = 5 I obtained,

[5a] $-2x(-1+z)(y+z^2) + yz(y+z^2) + x(y+2z^2) + (1-2x-2(-1+y)(-1+z)+(-1+z)^2)(x+2yz+z^3) + 2(x^2+yz^2(2y+z^2)+2x(2yz+z^3))$

After collecting terms this is simplified to

[5b] S(5) = $5y^2z + 5yz^3 + z^5 + 5x(y+z^2)$

As another example of a polynomial I will describe below,

[6] S(13) = $13x^4z + 13y^6z + 91y^5z^3 + 182y^4z^5 + 156y^3z^7 + 65y^2z^9 + 13yz^{11} + z^{13} + 13x^3(2y^2 + 10yz^2 + 5z^4) + 13x^2(10y^3z + 30y^2z^3 + 21yz^5 + 4z^7) + 13x(y^5 + 15y^4z^2 + 35y^3z^4 + 28y^2z^6 + 9yz^8 + z^{10})$

Some features of this growing polynomial with n are; (1) after n = 5, two new monomials are added for each increasing n. In the above example $13x^3(2y^2)$ and $13x(y^5)$ are new monomial terms from S(12), (2) the remaining terms are increasing in powers by one. For example the z^5 term in S(5) increased to z^{13}, the $5xz^2$ term in S(5) increased to $13xz^{10}$ in S(13) and (3) the coefficients of each monomial are increasing by an integer sequence. For example the coefficients of terms $5y^2z$ and $65y^2z^9$ in S(5) and S(13), respectively follow OEIS A000096 where coefficient = n*(n-3)/2 for the S(n) term.

Calculus of Intersequence Polynomials (ISPs)

As shown above in [5] and [6], multivariate polynomials can represent using equation [4] any sequence of the cubic polynomial $x^3 – c_2 x^2 – c_1 x – c_0 = 0$. Since the sequence S(n) is a function of the variables x, y and z these equations are amenable to integration and differentiation. Graphical descriptions of S(n) can also be shown as a function of x, y or z as well as in two and three dimensions. The integer values of S(n) are maintained if we use integer values of c_2, c_1, and c_0 but the equations are continuous, and any real value can be substituted for these coefficients. The equations are inter-sequential since they represent variation between different sequences. The variation in x, y and z are variations in the values of c_2, c_1, and c_0. If these c coefficients are kept constant, then only n can be varied, and this variation is intra-sequence. It can be shown that n must remain an integer since the series functions F7 and F8 depend on integers only. If functions are eventually found to be continuous in x, y, z and n (e.g three-dimensional Legendre polynomials) then an intra-sequence calculus could also be described.

Let S_n(x,y,z) be the nth ISP. All series values for S_{13}(x,y,z) can be easily calculated from [6] for any value of (x, y, z). Examples are $S_{13}(1,1,0) = 39$, $S_{13}(1,0,1) = 144$, $S_{13}(1,1,1) = 2757$, $S_{13}(0,1,1) = 521$ for the Perrin, the related Narayana cows, Tribonacci, and Lucas sequences, respectively. Partial derivatives can be found when setting two variables as constants. Given the polynomial S_{13}(x,y,z) in equation [6] the following partial derivatives are found.

[7a] $\partial_x S_{13}(x,1,1) = 13(89 + 130x + 51x^2 + 4x^3)$

[7b] $\partial_y S_{13}(1,y,1) = 13(41 + 130y + 171y^2 + 116y^3 + 40y^4 + 6y^5)$

[7c] $\partial_z S_{13}(1,1,z) = 13(12 + 50z + 111z^2 + 160z^3 + 175z^4 + 168z^5 + 112z^6 + 72z^7 + 45z^8 + 10z^9 + 11z^{10} + z^{12})$

Substituting x=0, y= 0 and z= 0 above provides slopes for the Lucas, the related Narayana cows, and Perrin sequences. For the Tribonacci sequence values for (x,y,z) which are (1,1,1), the partial derivatives at n = 13 are found. These values are the limiting slopes in the x, y, and z directions, respectively.

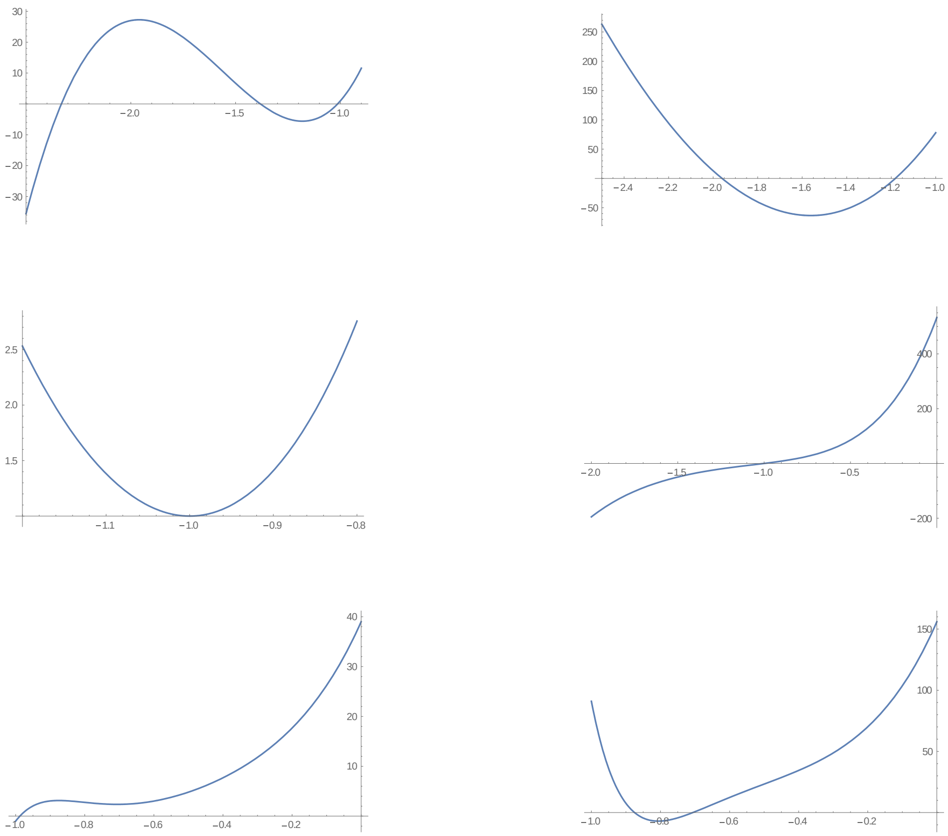
[8a] $\partial_x S_{13}(1,1,1) = 3562$

[8b] $\partial_y S_{13}(1,1,1) = 6552$

[8c] $\partial_z S_{13}(1,1,z) = 12051$

Note again that any value of (x, y, z) can be used in equation [6] to find the value of the derivative as in the example above. With integer coefficients the value will always be an integer. Mixed partial derivatives can also be calculated for $\partial_{x,y}\ \partial_{x,z}\ \partial_{y,z}$ *and* $\partial_{x,y,z}$.

Derivatives can locate maxima and minima of the ISP functions S_n(x,y,z). In the graphs below the one-dimension polynomials are shown on the left with the corresponding first derivative at right. Derivative values at zero correspond to local maxima and minima in the curves. They are shown to be the limiting slopes at each point on S_n(x,y,z).



Top to bottom left: $S_{13}(x, 1, 1)$, $S_{13}(1, y, 1)$, $S_{13}(1, 1, z)$. Top to bottom right: $\partial_x S_{13}(x,1,1)$, $\partial_y S_{13}(1,y,1)$, $\partial_z S_{13}(1,1,z)$

From these curves and appropriate root locator software the values of $S_{13}(x,y,z)$ at the minima and maxima are found. Values of the S function are easily found, e.g. $S_{13}(-1,1,1) = 1$ and $S_{13}(1,1,1) = 2757$.

The integration of these polynomials is also straightforward. Definite integrals are defined for all n and $S_n(x,y,z)$. We have the following integrals,

[9a] $\int_a^b S_n(x,y,z)\,dx$

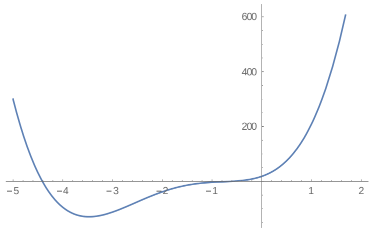
[9b] $\int_a^b S_n(x,y,z)\,dy$

[9c] $\int_a^b S_n(x,y,z)\,dz$

Multiple integrals of the three variables are also allowed. As an example, we find that the above functions for $S_{13}(x,y,z)$ are not symmetric about the x, y or z axis. If we form the double integral;

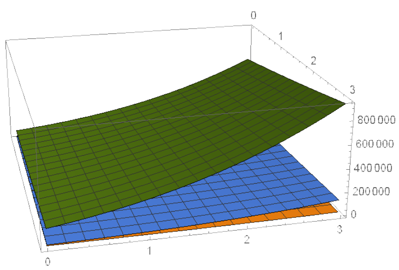
[10] $\int_0^1 \int_0^1 S_{13}(x,y,z)\,dz\,dy = \frac{2213}{120} + \frac{2769x}{44} + 78x^2 + \frac{130x^3}{3} + \frac{13x^4}{2}$

The function in x is plotted below between -5 and 2 and the area is found as $\frac{285719}{660} = 432.907\overline{57}$



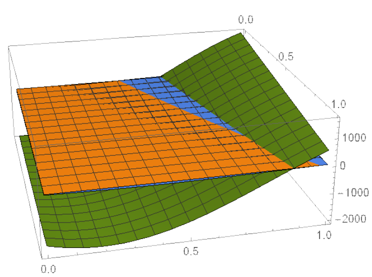
Plot of the doubly integrated function $S_{13}(x, y, z)$ in [10]

The above integrations obey the first fundamental theorem of calculus. If $S_n(x,y,z)$ is continuous on the closed interval [a,b] and F is the indefinite integral $\int S_n(x, y, z)\,dx$ on any continuous interval, then $\int_a^b S_n(x, y, z)\,dx = F(b) - F(a)$ true for x, y or z. In the figure below we find the surface planes of n = 10 for three functions of $S_{10}(x,y,z)$ for constant z and variable x and y.



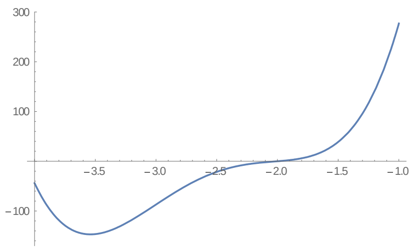
Surface plots of $S_{10}(x, y, 1)$, $S_{10}(x, y, 2)$, $S_{10}(x, y, 3)$ (bottom to top sheet) between 0>x<3 and 0>y<3

The surface sheets can intersect each other (see plot below)



Surface plots of $S_{10}(x, -1, z)$, $S_{10}(x, -2, z)$, $S_{10}(x, -4, z)$ (orange, blue, green) between 0>x<1 and 0>z<1

A plot of $S_{10}(1, y, 2)$ between -1>y<-4 shows how the y curve can dip below zero and result in a negative integration.



The function $S_{10}(1, y, 2)$. The integral between -4 and -1 is exactly -78.

Unit transitions between surfaces lead to new sequences. Consider the following polynomials $S_n(1,0,0)$, $S_n(0,1,0)$, $S_n(0,0,1)$. Respectively these are OEIS A021337 (decimal expansion terms of 1/333, $0\bar{0}3$), OEIS A010673 (decimal expansion terms of 1/495, $\bar{0}2$), and OEIS A057427 (decimal expansion terms of 1/90, $\bar{1}1$). Adding 1 to each polynomial expression yields a new surface; $S_n(1,1,0)$, $S_n(0,1,1)$, $S_n(1,0,1)$ representing the Perrin sequence, the Lucas sequence and the related Narayana cows sequence.). Adding 1 to each polynomial expression yields the same new surface; $S_n(1,1,1)$, $S_n(1,1,1)$, $S_n(1,1,1)$, the Tribonacci-like sequence (OEIS A001644- 3,1,3,7,11,21..). Further addition of 1 to these sequence result in an infinite number of new sequences and surfaces. The connection of these surfaces to applications in combinatorics and orthogonality is still to be explored.

Richard Turk

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Table of the First 12 ISPs $S_n(x,y,z)$

n	ISP for Sequence term $S_n(c_0, c_1, c_2)$ of cubic polynomial $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$
2	$2y + z^2$
3	$3x + 3yz + z^3$
4	$2y^2 + 4xz + 4yz^2 + z^4$
5	$5y^2z + 5yz^3 + z^5 + 5x(y + z^2)$
6	$3x^2 + 2y^3 + 9y^2z^2 + 6yz^4 + z^6 + 6x(2yz + z^3)$
7	$7x^2z + 7y^3z + 14y^2z^3 + 7yz^5 + z^7 + 7x(y^2 + 3yz^2 + z^4)$
8	$2y^4 + 16y^3z^2 + 20y^2z^4 + 8yz^6 + z^8 + 4x^2(2y + 3z^2) + 8x(3y^2z + 4yz^3 + z^5)$
9	$3x^3 + 9y^4z + 30y^3z^3 + 27y^2z^5 + 9yz^7 + z^9 + 9x^2(3yz + 2z^3) + 9x(y^3 + 6y^2z^2 + 5yz^4 + z^6)$
10	$2y^5 + 10x^3z + 25y^4z^2 + 50y^3z^4 + 35y^2z^6 + 10yz^8 + z^{10} + 5x^2(3y^2 + 12yz^2 + 5z^4) + 10x(4y^3z + 10y^2z^3 + 6yz^5 + z^7)$
11	$11y^5z + 55y^4z^3 + 77y^3z^5 + 44y^2z^7 + 11yz^9 + z^{11} + 11x^3(y + 2z^2) + 11x^2(6y^2z + 10yz^3 + 3z^5) + 11x(y^4 + 10y^3z^2 + 15y^2z^4 + 7yz^6 + z^8)$
12	$3x^4 + 2y^6 + 36y^5z^2 + 105y^4z^4 + 112y^3z^6 + 54y^2z^8 + 12yz^{10} + z^{12} + 8x^3(6yz + 5z^3) + 6x^2(4y^3 + 30y^2z^2 + 30yz^4 + 7z^6) + 12x(5y^4z + 20y^3z^3 + 21y^2z^5 + 8yz^7 + z^9)$
13	$13x^4z + 13y^6z + 91y^5z^3 + 182y^4z^5 + 156y^3z^7 + 65y^2z^9 + 13yz^{11} + z^{13} + 13x^3(2y^2 + 10yz^2 + 5z^4) + 13x^2(10y^3z + 30y^2z^3 + 21yz^5 + 4z^7) + 13x(y^5 + 15y^4z^2 + 35y^3z^4 + 28y^2z^6 + 9yz^8 + z^{10})$