Higher Order Inter Sequence Polynomials (ISPs)

Inter sequence polynomials of a general cubic monic polynomial were discussed in previous chapters which were derived from the cubic equation $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$ and previously expressed as linear recurrences,

[1] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = S_n(x,y,z)$

In this chapter I discuss fourth order ISPs defined for the monic quartic equation $\mathbf{x^4 - q_3\, x^3 - q_2\, x^2 - q_1\, x - q_0 = 0}$ where the linear recurrence is expressed as,

[2] $S_n(\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = S_n(w,x,y,z)$

These polynomials (see Table below) express powers of n in the Diophantine equation,

[3] $S_n(\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = \psi_1^n + \psi_2^n + \psi_3^n + \psi_4^n$

where n = 0, 1, 2, …. N and ψ_i are solutions to the quartic equation where $\mathbf{w = q_0}$, $\mathbf{x=q_1}$, $\mathbf{y = q_2}$, $\mathbf{z = q_3}$.

Like the cubic case these ISPs are true for real and complex values of ($\mathbf{q_0}$, $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$). As shown previously, these ISPs are continuous polynomials in variables w, x, y and z, they have partial derivatives and can be integrated.

Before discussing the properties of these polynomials, I will discuss one application of the cubic ISPs. When looking for roots of 4th order equations, a resolvent cubic is defined from coefficients of the monic quartic equation. Finding roots of the resolvent cubic leads to equations useful for finding the roots of the quartic equation. Define three equations expressing the cubic equation in terms of the coefficients from the quartic:

[3a] $c_2 = -q_2$

[3b] $c_1 = q_1 * q_3 - 4\, q_0$

[3c] $c_0 = 4\, q_0 * q_2 - q_1^2 - q_0 * q_3^2$

The corresponding "resolvent cubic" $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$ can be solved using root finding methods to find roots ψ_1, ψ_2, ψ_3. These roots have the following relation to the roots of the quartic $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

[4a] $\psi_1 = \alpha_1 * \alpha_2 + \alpha_3 * \alpha_4$

[4b] $\psi_2 = \alpha_1 * \alpha_3 + \alpha_2 * \alpha_4$

[4c] $\psi_3 = \alpha_1 * \alpha_4 + \alpha_2 * \alpha_3$

With one extra relation which is true for all linear recurrences,

[5] $-q_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$

these four equations with the LHS values known can be solved with root finders for simultaneous equations. A total of 7 roots are required to solve the quartic by this method. However, using cubic ISPs only the roots of the quartic need to be found. This is because we can find the sum of powers of equations [4a] to [4c]. Define the 3 equations:

[6] $B_n = (\alpha_1 * \alpha_2 + \alpha_3 * \alpha_4)^n + (\alpha_1 * \alpha_3 + \alpha_2 * \alpha_4)^n + (\alpha_1 * \alpha_4 + \alpha_2 * \alpha_3)^n = \psi_1^n + \psi_2^n + \psi_3^n$

with n = 1,2,3. Then B_n are the first three entries in the table below (z, $2y + z^2$, $3x + 3yz + z^3$) with $x = c_0$, $y = c_1$ and $z = c_2$. So, we can find the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ from the three equations [6] and equation [5] **without** a knowledge of ψ_1, ψ_2, ψ_3. *Mathematica* will easily find the four roots from [5] and [6] using "**NSolve**". Once these roots are found then the powers of these roots can be calculated on the RHS of [3] to find the sequence of the original quartic equation, $\mathbf{x^4 - q_3 x^3 - q_2 x^2 - q_1 x - q_0}$.

But, the same information and the LHS of [3] can be obtained by knowing the quartic ISPs! Comparing the first 3 entries in the Table below we find that these are exactly the cubic ISPs. Things start to change as n increases. Why?

The cubic ISPs were obtained from two linear recurrences of three variables that I labeled as P7 and P8. These recurrences are found in Chapter 34, equations [34-2] to [34-4]. Although, any cubic ISP can be calculated from a given n using this formula, and a formula could be developed for quartic ISPs, instead one can use the recurrence relation;

[7] $S_n(w,x,y,z) = \mathbf{z}*S_{n-1}(w,x,y,z) + \mathbf{y}*S_{n-2}(w,x,y,z) + \mathbf{x}*S_{n-3}(w,x,y,z) + \mathbf{w}*S_{n-4}(w,x,y,z)$

Yes, there was an easier path to get here!

Since $S_0(w,x,y,z) = 4$ from equation [3] above we can calculate the n = 4th term from the table below;

[8] $S_4(w,x,y,z) = \mathbf{z}*S_3(w,x,y,z) + \mathbf{y}*S_2(w,x,y,z) + \mathbf{x}*S_1(w,x,y,z) + \mathbf{w}*S_0(w,x,y,z)$

$= \mathbf{z}*(3x + 3yz + z^3) + \mathbf{y}*(2y + z^2) + \mathbf{x}*z + \mathbf{w}*4 = 4w + 2y^2 + 4xz + 4yz^2 + z^4$

Higher terms in n can be easily calculated from [7]. The Table shows the first 12 terms as functions of w, x, y, and z.

These ISPs represent sequences of order 1, 2, 3 and 4 derived from monic, quadratic, cubic and quartic equations. It can be shown that cubic equations are represented properly as ***S_n(1,1,0) = S_n(0,1,1,0)*** or ***S_n(1,1,1) = S_n(0,1,1,1).*** Some well known quartic sequences can be described by ***S_n(1,1,1,1)*** (OEIS A073817 Tetranacci numbers), and Fielder sequences ***S_n(1,0,1,1)*** (OEIS A001641) and ***S_n(1,1,0,1)*** (OEIS A001638).

Derivative properties are similar for quartic polynomials. For example, we can show that the Perrin and Lucas numbers obey:

[9a, b, c] $\partial_x S_n(0,1,1,0) = n * Pd(n)$, $\partial_y S_n(0,1,1,0) = (n+1) * Pd(n+1)$, $\partial_z S_n(0,1,1,0) = (n+2) * Pd(n+2)$,

[10a, b, c] $\partial_x S_n(0,0,1,1) = n * F(n)$, $\partial_y S_n(0,0,1,1) = (n+1) * F(n+1)$, $\partial_z S_n(0,0,1,1) = (n+2) * F(n+2)$

Some new derivatives are

[11] $\partial_z S_n(1,1,0,1) = (n+2) * A006498(n+2)$,

[12] $\partial_z S_n(1,1,1,1) = (n+2) * A000078(n+2)$, and $\partial_{zz} S_n(1,1,1,1) = (n+2) * A118898(n+2)$

Although fewer named quartic sequences are found, the element sequences are found from the first derivative. The following definition applies to quartic ISPs and possibly to higher order ISPs as well.

*Definition- The **element** sequence $\mathcal{E}(\mathbf{n})$ is obtained from the1st derivative of the sequence S_n(**q_0, q_1, q_2, q_3**): $\partial_z S_n(\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}) = \boldsymbol{n} * \mathcal{E}(\mathbf{n})$*

Theorem: The derivatives of the sequence S_n(q_0, q_1, q_2, q_3) are convolution operators on the element sequence $\mathcal{E}$(n).

As an example

[13] $\partial_z S_n(1,1,1,1) = n * \mathcal{E}(\mathrm{n}) = n * A000078(n) = n * \{1.,1.,2.,4.,8.,15.,29.,56.,108 ....$

[14] $(\mathcal{E} * \mathcal{E})(\mathrm{n}) = \sum_{j=1}^{n} \mathcal{E}(\mathrm{j})\mathcal{E}(n+1-j) = \mathcal{E}_1(\mathrm{n}) \rightarrow \partial_{zz} S_n(1,1,1,1) = [n\, *]A118898(\mathrm{n}) = [\mathrm{n}*]$
$\{0.,1.,2.,5.,12.,28.,62.,136.,294 ....$

Another observation regarding derivatives of linear sequences is found in the generating polynomial for the sequence. The generating function for the first derivative of the Perrin sequence expresses the Padovan sequence:

[15] $1/(1-x^2-x^3) = 1 + 0*x^1 + 1*x^2 + 1*x^3 + 1*x^4 + 2*x^5 + 2*x^6 + 3*x^7 + \cdots$

Higher derivatives can be expressed by this generating function. The n^{th} derivative agrees with the sequence $\mathcal{E}_{n-1}(\mathrm{n})$ and the generating function,

[16] $(-1)^{n-1}\, x^{n-2}/(1-x^2-x^3)^n$

Where the numerator x, shifts the sequence to the right. For example, the fourth derivative is expressed by:

[17] $(-1)^{n-1}x^2/(1-x^2-x^3)^4 = 0 + 0*x^1 + 1*x^2 + 0*x^3 + 4*x^4 + 4*x^5 + 10*x^6 + 20*x^7 + 30*x^8 + \cdots$

Returning to the roots of a quartic, knowing the quartic ISPs (for n= 1 , n = 2 and n = 3) provides three equations for a numerical solution to any quartic equation:

[18] $BQ_n = (\alpha_1)^n + (\alpha_2)^n + (\alpha_3)^n + (\alpha_4)^n =$ ***S_n(q_0, q_1, q_2, q_3)***

The fourth equation is a property of our cubic and quartic equations;

[19] $BQ_4 = (\alpha_1)^1 * (\alpha_2)^1 * (\alpha_3)^1 * (\alpha_4)^1 = Q_0$

ISPs also define various polynomial sequences found in the literature. For example, the Lucas Polynomial sequences can be expressed by ISPs as a function of the variable x.[1] Let q_2 = q(x) and q_3 = p(x) then the following polynomials W(x) and w(x) are defined from reference [1].

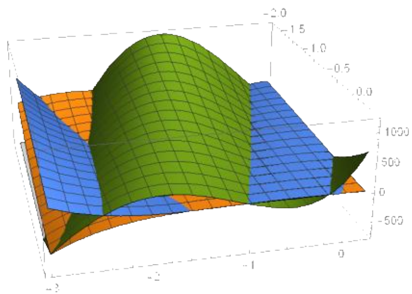
p(x)	q(x)	Polynomial W(x)	$\partial_z S_n(0,0,\mathrm{q(x)},\mathrm{p(x)})/n$	Polynomial w(x)	$S_n(0,0,\mathrm{q(x)},\mathrm{p(x)})$
x	1	Fibonacci	$\partial_z S_n(0,0,1,\mathrm{x})/n$	Lucas	$S_n(0,0,1,\mathrm{x})$
2x	1	Pell	$\partial_z S_n(0,0,1,2\mathrm{x})/n$	Pell-Lucas	$S_n(0,0,1,2\mathrm{x})$
1	2x	Jacobsthal	$\partial_z S_n(0,0,2\mathrm{x},1)/n$	Jacobsthal-Lucas	$S_n(0,0,2\mathrm{x},1)$
3x	-2	Fermat	$\partial_z S_n(0,0,-2,3\mathrm{x})/n$	Fermat-Lucas	$S_n(0,0,-2,3\mathrm{x})$
2x	-1	Chebyshev 2nd kind	$\partial_z S_n(0,0,-1,2\mathrm{x})/n$	2*Chebyshev 1st kind	$S_n(0,0,-1,2\mathrm{x})$

[1] Lucas Polynomial Sequence from Wolfram Mathematica-mathworld.wolfram.com

x+1	-x	$\frac{(x^n - 1)}{x-1}$	$\partial_z S_n(0,0,-\text{x},\text{x}+1)/n$	$x^n + 1$	$S_n(0,0,-\text{x},\text{x}+1)$

These polynomials are easily expressed by the ISP or the first derivative of the ISP divided by n. Many other polynomial sequences can be expressed such as Perrin and Padovan Polynomial sequences; $S_n(0,1,\text{x},0)$ and $\partial_z S_n(0,1,\text{x},0)/n$ respectively.

The quartic ISPs represent 4 dimensional surfaces for given values of n≥4. A look at some surfaces at n = 13 demonstrates how these sheets are interwoven with continuity;



Surface plots of $S_{13}(0, x, y, 0)$, $S_{13}(-1, x, y, 1)$, $S_{13}(-2, x, y, 2)$ (orange, blue, green sheets) between -2>x<1/4 and -3>y<1/4

In closing this subject there are some questions to consider:

1. Can the derivatives of $S_n(\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3})$ be used to find solutions to the quartic?
2. Do derivatives of $S_n(\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3})$ define differential equations?
   A quick note that the answer is **Yes**. See reference[2]. Consider the equation y(x)'' – 2y(x)'- y(x) = 0. A solution is y(x) = $(1/a_1)*(e^{\phi 1 * x} - e^{\phi 2 * x})$ with ϕ1 and ϕ2 solutions to the equation $z*(z^2 – 2z -1)= 0$ and a_1 a constant. Then the first derivative of the ISP $S_n(0,0,1,2)$ gives the element sequence $\mathcal{E}_0(\text{n})$ ={1,2,5,12,29,70,169,408,985,2378 … }. It can be shown that a series solution y(x) = $(1/(2\sqrt{2})) * (\sum_{n=1}^{\infty} (\phi 1 * x)^n/n! - \sum_{n=1}^{\infty} (\phi 2 * x)^n/n!)$ is equivalent to the series y(x) = $\sum_{n=1}^{\infty} \mathcal{E}_0(\text{n}) * (x)^n/n!$ and y(x) = $(1/(2\sqrt{2}))*(e^{\phi 1 * x} - e^{\phi 2 * x})$ where all y(x) satisfy the original differential equation! Let Y[n] = $\sum_{n=1}^{\infty} \mathcal{E}_0(\text{n}) * (x)^n/n!$ be the sequence of coefficients for powers of x. Then the following sequence equation is true for any second order differential equation y(x)'' – Ay(x)'-By(x) = 0 , $Y[[n+1]] * (n) * (n-1) - A * Y[[n+1]] * (n) - B * Y[[n]] = \{0,0,0,0,\ldots\}$
3. Can FFT fast Fourier transforms be used for solution?
4. Are there other hidden operations like convolution which can be uncovered using ISPs?
5. Is there a subtle link between ISPs and Diophantine equations of higher order?

Perhaps we could have taken a simpler path to this point regarding linear sequences. We found continuous functions describing a discrete set of numbers and we discovered a new operation for finding element sequences. We visualized the sequence surface sheets and their derivatives. I hope we learned some new ideas using ISPs and anticipate they will uncover other new aspects of linear sequences by looking at their geometry.

[2] M. Waldschmidt, Linear Recurrence Sequences: an Introduction, ICSFA-2018 November 22-24, 2018.

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Table of the First 12 Quartic ISPs $S_n(w,x,y,z)$

n	ISP for Sequence term $S_n(q_0, q_1, q_2, q_3)$ of quartic polynomial $\mathbf{x^4 - q_3 x^3 - q_2 x^2 - q_1 x - q_0 = 0}$
1	z
2	$2y + z^2$
3	$3x + 3yz + z^3$
4	$4w + 2y^2 + 4xz + 4yz^2 + z^4$
5	$5x(y + z^2) + z(5w + 5y^2 + 5yz^2 + z^4)$
6	$3x^2 + 2y^3 + 9y^2z^2 + 6yz^4 + z^6 + 6w(y + z^2) + 6x(2yz + z^3)$
7	$7x^2z + 7y^3z + 14y^2z^3 + 7yz^5 + z^7 + 7w(x + 2yz + z^3) + 7x(y^2 + 3yz^2 + z^4)$
8	$4w^2 + 2y^4 + 16y^3z^2 + 20y^2z^4 + 8yz^6 + z^8 + 4x^2(2y + 3z^2) + 8w(y^2 + 2xz + 3yz^2 + z^4) + 8x(3y^2z + 4yz^3 + z^5)$
9	$3x^3 + 9x^2(3yz + 2z^3) + 9x(2wy + y^3 + 3wz^2 + 6y^2z^2 + 5yz^4 + z^6) + z(9w^2 + 9y^4 + 30y^3z^2 + 27y^2z^4 + 9yz^6 + z^8 + 9w(3y^2 + 4yz^2 + z^4))$
10	$2y^5 + 10x^3z + 25y^4z^2 + 50y^3z^4 + 35y^2z^6 + 10yz^8 + z^{10} + 5w^2(2y + 3z^2) + 5x^2(3y^2 + 12yz^2 + 5z^4) + 10w(x^2 + y^3 + 6xyz + 6y^2z^2 + 4xz^3 + 5yz^4 + z^6) + 10x(4y^3z + 10y^2z^3 + 6yz^5 + z^7)$
11	$11y^5z + 55y^4z^3 + 77y^3z^5 + 44y^2z^7 + 11yz^9 + z^{11} + 11x^3(y + 2z^2) + 11w^2(x + 3yz + 2z^3) + 11x^2(6y^2z + 10yz^3 + 3z^5) + 11x(y^4 + 10y^3z^2 + 15y^2z^4 + 7yz^6 + z^8) + 11w(3x^2z + 4y^3z + 10y^2z^3 + 6yz^5 + z^7 + x(3y^2 + 12yz^2 + 5z^4))$
12	$4w^3 + 3x^4 + 2y^6 + 36y^5z^2 + 105y^4z^4 + 112y^3z^6 + 54y^2z^8 + 12yz^{10} + z^{12} + 8x^3(6yz + 5z^3) + 6w^2(3y^2 + 6xz + 12yz^2 + 5z^4) + 6x^2(4y^3 + 30y^2z^2 + 30yz^4 + 7z^6) + 12x(5y^4z + 20y^3z^3 + 21y^2z^5 + 8yz^7 + z^9) + 12w(y^4 + 10y^3z^2 + 15y^2z^4 + 7yz^6 + z^8 + 3x^2(y + 2z^2) + 2x(6y^2z + 10yz^3 + 3z^5))$