Inter Sequence Polynomials (ISPs) and Fermat's Last Theorem

The inter sequence polynomials of a general cubic monic polynomial were generated from the cubic equation $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$ and expressed as

[1] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = S_n(x,y,z)$

Fermat's last theorem asserts that for powers of n,

[2] $x^n + y^n = z^n$

has integer solutions for n = 2 but there are no integer solutions for n>2. From [1] we showed in the previous chapter that

[3] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = \psi^n + \psi_1^n + \psi_2^n$

where n = 0, 1, 2, …. N and ψ_i are solutions to the cubic equation.

For n = 2 the inter sequence polynomial is $2y + z^2$ or $2\,\mathbf{c_1} + \mathbf{c_2}^2$ for any c_1 and c_2. Note that this equation does not depend on the c_0 coefficient. As an example of a known solution to [2] with n = 2, consider (ψ_1, ψ_2, ψ_3) = (28, 45, 53). These are solutions to the cubic polynomial;

[4] $(x-28)(x-45)(x-53) = -66780 + 5129x - 126x^2 + x^3$

with $\mathbf{c_0}$ = 66780, $\mathbf{c_1}$ = -5129, and $\mathbf{c_0}$ = 126.

Calculate $S_2(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ to obtain the result from [3]

[5] $S_2(66780, -5129, 126) = 5618 = 28^2 + 45^2 + 53^2$

and as known,

[6] $28^2 + 45^2 = 53^2$

If instead we take only 2 solutions to [4] and assume the third solution is zero then,

[7] $(x-28)(x-45)(x) = 1260x - 73x^2 + x^3$

and the inter sequence polynomial is

[8] $S_2(0, -1260, 73) = 2809 = 53^2$

The new polynomial finds the second term for a Lucas type sequence, $S_n(0,-y,z)$.

Note that z = 73 = 28 + 45 and y= 1260 = 28*45. We expect this to be true for any two solutions (u,v) but only certain combination will result in a square of an integer as in [8]. Another observation is that c_0 from the Tribonacci type sequence [5] when divided by c_1 from the Lucas type sequence is 66780/1260 = 53.

Note also that the (x,y,z) values in sequences [5] and [8] are the same for **all** values of n. Let n = 3 and using the same cubic polynomial in [4]

[9] $S_3(66780, -5129, 126) = 261954 = 28^3 + 45^3 + 53^3$

where $S_3(x, y, z) = 3x + 3yz + z^3$

As expected, $\boldsymbol{28^3 + 45^3 \neq 53^3}$ even though [9] is correct.

If we use the Lucas equation,

[10] $S_3(0, -1260, 73) = 13077 = \boldsymbol{28^3 + 45^3} = (48.35685999\ldots)^3$

again, showing that the solution is not a cube of an integer. Since [9] and [10] are true for any n with u = 28 and v = 45 we do not expect to find an integer solution for the root of $(S_n(66780, -5129, 126))^{(1/n)}$.

If we let (ψ_1, ψ_2, ψ_3) = (u,v,w) as in [8], we find for all inter sequence polynomials that

[11] $S_n(0, -u*v, (u+v)) = \boldsymbol{u^n + v^n}$

Let $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$ and (ψ_1, ψ_2, ψ_3) = (u,v,w) then as in [5]

[12] $S_n(u*v*w, -(u*v+u*w+v*w), (u+v+w)) = \boldsymbol{u^n + v^n + w^n}$

To give a demonstration of Fermat's last theorem using inter sequence polynomials, I show that [11] can be reduced to a square for certain values of u and v. This demonstration is based on that of Hardy and Wright.[1] First notice that for n = 2 the parity of the solution requires that one number (u) is odd and v is even. This is handled by defining

[13] $v = 2ab$ and $u = a^2 - b^2$

Then for every value of n from [11] and the inter sequence polynomials we have

[14] $y = -(2ab)*(a^2 - b^2)$ and $z = (a^2 - b^2) + 2ab$

Starting with n = 2

[15] $2y + z^2 = 2(-2a * b) * (a^2 - b^2) + (2a * b + a^2 - b^2)^2 = a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2$

From the example above [8] we find a = 7 and b = 2 with y = -1260 and z = 73

The above example shows that a square can be generated from 3 numbers represented by integers a and b.

For n = 3 the inter sequence polynomial for x = 0 (See Table below) is;

[16] $3yz + z^3 = 3(-2a * b * (a^2 - b^2))(2a * b + a^2 - b^2) + (2a * b + a^2 - b^2)^3$

$= a^6 - 3a^4b^2 + 8a^3b^3 + 3a^2b^4 - b^6 = (a^2 + 2ab - b^2)(a^4 - 2a^3b + 2a^2b^2 + 2ab^3 + b^4)$

In this case a and b are not factored into a cube for any integer value. The same is true for n = 4

[17] $2y^2 + 4yz^2 + z^4 = a^8 - 4a^6b^2 + 22a^4b^4 - 4a^2b^6 + b^8$

This expression cannot be factored further so it cannot be a 4th power of two integers a and b.

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[1] G. H. Hardy and E. M Wright, An Introduction to the Theory of Numbers, Oxford University Press, 6th Edition 2008, pp 245-247.

For n = 7 as another test I find the factored ISP as,

[18] $(a^2+2ab-b^2)(a^{12}-2a^{11}b-2a^{10}b^2+2a^9b^3+15a^8b^4-28a^7b^5+36a^6b^6+28a^5b^7+15a^4b^8-2a^3b^9-2a^2b^{10}+2ab^{11}+b^{12})$

It remains to be shown that all ISPs above n=2 cannot be factored into the nth power of integers a and b. This demonstrates that a form of solution of equation [2] based on equation [14] verifies Fermat's theorem.

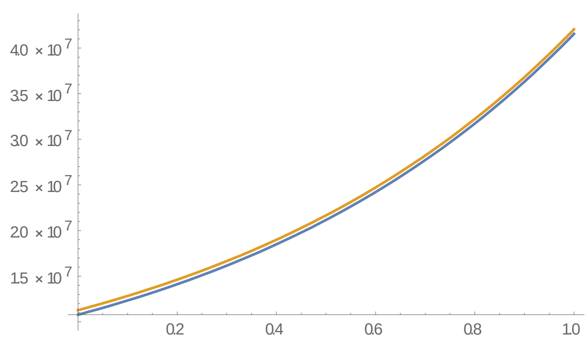
This Chapter will serve as a guide for showing new properties of Fermat's last theorem as well as other Diophantine equations using the inter sequence polynomials.

Curve fitting and Polynomial mixing

As mentioned previously the inter sequence polynomials are functions of x, y, and z and are monomial forms representing the nth sequence term where n is an integer. Equation [2] is true for all rational values of n and is not restricted to integers. If n is not an integer and rational, then the value of $S_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is not an integer but is a real number. From the ISPs we know the bounding integer values between $S_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $S_{n+1}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and it can be shown that $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ is a continuous function between two integer values. Polynomials are commonly used for curve fitting and it would be advantages to use the ISP's for curve fitting between the n^{th} and $(n+1)^{th}$ terms of any sequence. As an example use the 12^{th} and 13^{th} ISPs to fit the variation of n between 12 and 13 for $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$. The figure below approximates the curve $S(n) = \psi^n + \psi_1^n + \psi_2^n$ with the curve using ISP terms in the table below with a single added variable a;

[19] $\mathrm{F1213[x,y,z,a]} = (3x^4+2y^6+36y^5z^2+105y^4z^4+112y^3z^6+54y^2z^8+12yz^{10}+z^{12}+8x^3(6yz+5z^3)+6x^2(4y^3+30y^2z^2+30yz^4+7z^6)+12x(5y^4z+20y^3z^3+21y^2z^5+8yz^7+z^9))^{1-a}(13x^4z+13y^6z+91y^5z^3+182y^4z^5+156y^3z^7+65y^2z^9+13yz^{11}+z^{13}+13x^3(2y^2+10yz^2+5z^4)+13x^2(10y^3z+30y^2z^3+21yz^5+4z^7)+13x(y^5+15y^4z^2+35y^3z^4+28y^2z^6+9yz^8+z^{10}))^a$

This equation is a mixing of the 12^{th} and 13^{th} ISP, set to a power **a** which is the fraction between 0 and 1 added to 12. We see the curves are indistinguishable although not exact,



Graph of $S(n) = \psi^n + \psi_1^n + \psi_2^n$ (blue) versus [19] (orange) for $S_n(c_0, c_1, c_2) = S_n(5,2,3)$ 13>n>12. Orange curve is displaced by $5*10^5$ to illustrate fit.

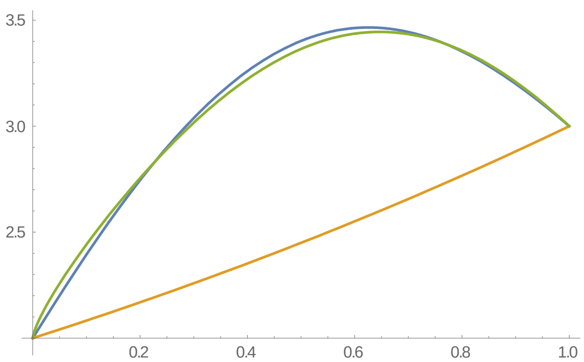
I find that the fit is very good for larger n values. In the example above, **a** is chosen such that n = 12 +a. For smaller n it is sometimes necessary to use powers of **a** to represent the curve. As an example, using the ISPs for the Perrin equation, $S_n(\mathbf{1}, \mathbf{1}, \mathbf{0})$ between the 2^{nd} and 3^{rd} term S(n) is not linear and curve fitting can be used on the equation,

[20] $$F23[x, y, z, a] = (2y + z^2)^{1-a}(3x + 3yz + z^3)^a$$

with adjustment of a values,

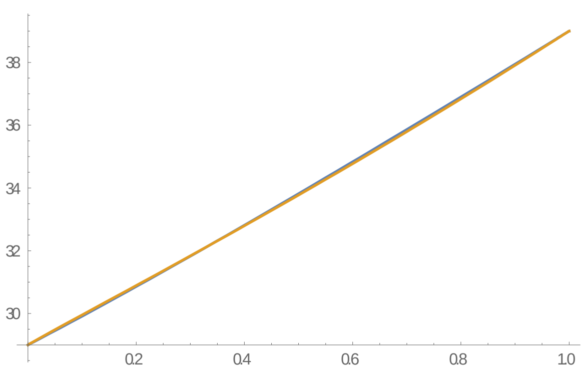
[21] $$F23a[x, y, z, a] = (2y + z^2)^{1-a^{2.3}}(3x + 3yz + z^3)^{a^{0.73}}$$

I obtain the curves;



Graph of S(n) = $\psi^n + \psi_1^n + \psi_2^n$ (blue) versus [20] (orange) and [21] (green) for $S_n(c_0, c_1, c_2) = S_n(1,1,0)$ 3>n>2. Note S(n) can have values above S(n+1).

Integrating [21] the maximum value of $F23a[1,1,0,a]$ occurs at 2.647. As n increases, good agreement is found for $S_n[1,1,0,a]$ versus S(n) with a relative error of only 0.0027% for n = 12+1/3 with the curve between 12 and 13 almost linear.



Graph of S(n) = $\psi^n + \psi_1^n + \psi_2^n$ (blue) versus [19] (orange)) for $S_n(c_0, c_1, c_2)$ = F1213(1,1, 0), 13>n>12.

As values of c_0, c_1, and c_2 increase, there is excellent agreement of the mixed polynomial curve fit with equation [3]. With good curve fitting software any function S(n) can be fitted within the desired precision using the ISPs for n and n+1. This fit requires that the roots ψ_ι in [3] are positive or complex. If ψ_ι is negative then a fractional power will result in an imaginary or complex value of $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ in [3]. In this case the magnitude of [3] and of the mixed polynomial curve are compared. It is possible that a parametric equation for the exponents of the variable **a** will eventually be found based on the values of x, y and z. The mixing of more than two ISPs has currently not been studied.

Derivatives of Integer Sequences Revisited

What is the derivative of an integer sequence? The ISPs give some meaning to this question. Previously, we found that the ISPs are continuous functions which can be differentiated and integrated. There is an interesting relation of a sequence to its derivatives. I show two examples; the derivative of the Perrin sequence and the derivative of the Lucas sequence. Using the table below to calculate derivatives of ***S_n(1,1,0)*** and ***S_n(0,1,1)*** which represent the Perrin and Lucas sequences, respectively I find the following derivative relationships

[22a, b, c] $\partial_x S_n(1,1,0) = n * Pd(n)$, $\partial_y S_n(1,1,0) = (n+1) * Pd(n+1)$, $\partial_z S_n(1,1,0) = (n+2) * Pd(n+2)$,

[23a, b, c] $\partial_x S_n(0,1,1) = n * F(n)$, $\partial_y S_n(0,1,1) = (n+1) * F(n+1)$, $\partial_z S_n(0,1,1) = (n+2) * F(n+2)$

where Pd(n) is the Padovan sequence (OEIS A000931) and F(n) is the Fibonacci sequence (OEIS A000045). In the above, the values of x, y and z are substituted after the derivative is taken. These are the sequences we originally demonstrated generate the Perrin and Lucas sequences. The successive derivatives shift the values of the generating sequence to the right. This property can also be shown for the Narayana sequence ***S_n(1,0,1)*** and the Tribonacci sequence ***S_n(1,1,1)***. The derivatives of these sequences are n times the associated generating sequence (Narayana cows (OEIS A000930) and the Tribonacci sequence (OEIS A000073). This derivative property is true for all sequences originating from roots of the cubic equation $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$. It remains to be shown whether the derivative properties of the inter sequence polynomials also provide new insight into the Diophantine equations.

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Table of the First 12 ISPs S_n(x,y,z)

n	ISP for Sequence term $S_n(c_0, c_1, c_2)$ of cubic polynomial $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$
2	$2y + z^2$
3	$3x + 3yz + z^3$
4	$2y^2 + 4xz + 4yz^2 + z^4$
5	$5y^2z + 5yz^3 + z^5 + 5x(y + z^2)$
6	$3x^2 + 2y^3 + 9y^2z^2 + 6yz^4 + z^6 + 6x(2yz + z^3)$
7	$7x^2z + 7y^3z + 14y^2z^3 + 7yz^5 + z^7 + 7x(y^2 + 3yz^2 + z^4)$
8	$2y^4 + 16y^3z^2 + 20y^2z^4 + 8yz^6 + z^8 + 4x^2(2y + 3z^2) + 8x(3y^2z + 4yz^3 + z^5)$
9	$3x^3 + 9y^4z + 30y^3z^3 + 27y^2z^5 + 9yz^7 + z^9 + 9x^2(3yz + 2z^3) + 9x(y^3 + 6y^2z^2 + 5yz^4 + z^6)$
10	$2y^5 + 10x^3z + 25y^4z^2 + 50y^3z^4 + 35y^2z^6 + 10yz^8 + z^{10} + 5x^2(3y^2 + 12yz^2 + 5z^4) + 10x(4y^3z + 10y^2z^3 + 6yz^5 + z^7)$
11	$11y^5z + 55y^4z^3 + 77y^3z^5 + 44y^2z^7 + 11yz^9 + z^{11} + 11x^3(y + 2z^2) + 11x^2(6y^2z + 10yz^3 + 3z^5) + 11x(y^4 + 10y^3z^2 + 15y^2z^4 + 7yz^6 + z^8)$
12	$3x^4 + 2y^6 + 36y^5z^2 + 105y^4z^4 + 112y^3z^6 + 54y^2z^8 + 12yz^{10} + z^{12} + 8x^3(6yz + 5z^3) + 6x^2(4y^3 + 30y^2z^2 + 30yz^4 + 7z^6) + 12x(5y^4z + 20y^3z^3 + 21y^2z^5 + 8yz^7 + z^9)$
13	$13x^4z + 13y^6z + 91y^5z^3 + 182y^4z^5 + 156y^3z^7 + 65y^2z^9 + 13yz^{11} + z^{13} + 13x^3(2y^2 + 10yz^2 + 5z^4) + 13x^2(10y^3z + 30y^2z^3 + 21yz^5 + 4z^7) + 13x(y^5 + 15y^4z^2 + 35y^3z^4 + 28y^2z^6 + 9yz^8 + z^{10})$