Inter Sequence Polynomials (ISPs), Linear Recurrence and Discrete Convolution

Inter sequence polynomials of a general cubic monic polynomial were derived from the cubic equation $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$ and previously expressed as linear recurrences,

[1] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = S_n(x,y,z)$

These polynomials (see Table below) express powers of n,

[2] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = \psi^n + \psi_1^n + \psi_2^n$

where n = 0, 1, 2, …. N and ψ_i are solutions to the cubic equation and $\mathbf{x = c_0}$, $\mathbf{y = c_1}$, $\mathbf{z = c_2}$.

The ISPs are true for real and complex values of ($\mathbf{c_0}$, $\mathbf{c_1}$, $\mathbf{c_2}$). Since ISPs are continuous polynomials in x, y and z, they also have partial derivatives and can be integrated.

The derivatives of ***$S_n(1,1,0)$*** and ***$S_n(0,1,1)$*** which represent the Perrin and Lucas sequences, have the following derivative relationships

[3a, b, c] $\partial_x S_n(1{,}1{,}0) = n * Pd(n)$, $\quad \partial_y S_n(1{,}1{,}0) = (n+1) * Pd(n+1)$, $\quad \partial_z S_n(1{,}1{,}0) = (n+2) * Pd(n+2)$,

[4a, b, c] $\partial_x S_n(0{,}1{,}1) = n * F(n)$, $\quad \partial_y S_n(0{,}1{,}1) = (n+1) * F(n+1)$, $\quad \partial_z S_n(0{,}1{,}1) = (n+2) * F(n+2)$

where Pd(n) is the Padovan sequence (OEIS A000931) and F(n) is the Fibonacci sequence (OEIS A000045). It can be shown that these relationships are generalized for any sequence obtained from a cubic equation $\mathbf{x^3 - c_2\, x^2 - c_1\, x - c_0 = 0}$. To specify this fact, I define the **element** sequence as $\mathcal{E}$(n) obtained from the ISP first derivatives of a sequence S_n(**x**, **y**, **z**). Then

[5] $\partial_i S_n(\mathrm{x,y,z}) = n * \mathcal{E}(\mathrm{n})$

where it is understood that i is a variable x, y or z and on the right side of [5] x = c0, y = c1 and z = c2 such that $\mathcal{E}$(n) is an element sequence. Examples of [5] are found in [3] and [4] above. For the remainder of this chapter the nth term of the sequence is defined as the value obtained from the ISP, $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ with the knowledge that the derivative with respect to x, y and z shift sequence terms to higher indices of n.

What of higher derivatives? If the index i remains the same variable then define the second derivative as,

[6] $\partial_{ii} S_n(\mathrm{x,y,z}) = n * \mathcal{E}_1(\mathrm{n})$

where $\mathcal{E}_1$(n) is another sequence. As we take more derivatives, I find that there is a general equation for the kth derivative,

[7] $\partial_{ii..i_k} S_n(\mathrm{x,y,z}) = (k-1)!\, n * \mathcal{E}_{k-1}(\mathrm{n})$

This equation is true for any sequence $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ where the coefficients can be real or complex! I show an example of [7] using the Lucas sequence. The ISPs start at n = 2 so terms 2 to 13 of the Lucas sequence are calculated from the ISP table as.

[8a] $S_n(0, 1, 1) = \{3.,4.,7.,11.,18.,29.,47.,76.,123.,199.,322.,521.\}$

[8b] $\partial_z S_n(0{,}1{,}1) = [n\,*]\{1.,2.,3.,5.,8.,13.,21.,34.,55.,89.,144.,233.\}$

[8c] $\partial_y S_n(0,1,1) = [n\,*]\{1.,1.,2.,3.,5.,8.,13.,21.,34.,55.,89.,144.\}$

[8d] $\partial_{zz} S_n(0,1,1) = 1! * [n\,*]\{1.,2.,5.,10.,20.,38.,71.,130.,235.,420.,744.,1308.\}$

[8e] $\partial_{zzz} S_n(0,1,1) = 2! * [n\,*]\{0.,1.,3.,9.,22.,51.,111.,233.,474.,942.,1836.,3522.\}$

[8f] $\partial_{zzzz} S_n(0,1,1) = 3! * [n\,*]\{0.,0.,1.,4.,14.,40.,105.,256.,594.,1324.,2860.,6020.\}$

In these equations the symbol "[n*]" indicates each term in the sequence is multiplied by the term number starting with n = 2 to n = 13. The sequences $\mathcal{E}_k$(n) clearly cannot be divided any further by [n*] so they are considered as reduced sequences. But what are the sequences $\mathcal{E}_k$(n)?

Let us look at a new operation on sequences. We know that sequences can be added, subtracted and multiplied. Another operation on sequences is convolution. Let f(x) and g(x) be two integer sequences with real or complex terms, and x represents the xth term. Then the *convolution* f*g is defined as;

[9] $(f*g)(x) = \sum_{y=1}^{x} f(y)g(x+1-y)$

This equation assumes that the sequence is indexed from the first term at y = 1 and can apply to any integer sequence of x terms. If f and g are the same sequence then (f*f)(x) is called the self-convolved sequence.

The above definition is known as linear convolution and in today's technology is a frequent computation used in the processing of digital signals. In signal processing a pulse sequence is convoluted with an input signal and the resulting convoluted signal is the output pulse. Convolution is also useful in statistical data analysis in which independent random variables are convoluted to find the probability distribution of their sum. In engineering, convolution and discrete Fourier analysis are commonly used mathematical tools for solving differential equations.

Let us apply convolution to the element sequence $\mathcal{E}$(n). I have shown in [8c] a right shifted form of [8b]. The sequence [8c] contains the y = 1 term which is needed for [9]. It is also useful to redefine the variable x as n and y as j to avoid confusion of the variables x, y, and z then the self- convolution of $\mathcal{E}$ is;

[10] $(\mathcal{E} * \mathcal{E})(n) = \sum_{j=1}^{n} \mathcal{E}(j)\mathcal{E}(n+1-j) = \mathcal{E}_1(n)$

where in my examples the upper limit of n is 12 resulting in 12 terms as n indexes from 1 to 12. We find from [8d] and [10]

[11] $\mathcal{E}_1(n) = \{1.,2.,5.,10.,20.,38.,71.,130.,235.,420.,744.,1308.\}$

The sequence $\mathcal{E}_1(n)$ is obtained by (1) self-convolution of the element sequence or (2) by the second derivative of the ISP for the Lucas sequence $S_n(0, 1, 1)$!

This result is somewhat of a paradox. The convolution of functions is usually defined by the integration of the functions f and g. For discrete convolution the integral is substituted with the summation as in [9]. But we see that convolution of [10] is equivalent to the result obtained from the second derivative of the ISP functions $S_n(x, y, z)$! What of higher derivatives? I find that

[12] $(\mathcal{E} * \mathcal{E}_1)(n) = \sum_{j=1}^{n} \mathcal{E}(j)\mathcal{E}_1(n+1-j) = \mathcal{E}_2(n)$

[13] $\mathcal{E}_2(n) = \{0.,1.,3.,9.,22.,51.,111.,233.,474.,942.,1836.,3522.\}$

Subsequent self-convolved sequences of the element sequence $(\mathcal{E}* \mathcal{E}_k)(n)$ results in the (k+1)st sequence obtained from the (k+2)nd derivative;

[14] $$(\mathcal{E}* \mathcal{E}_k)(n) = \mathcal{E}_{k+1}(n)$$

[15] $$(\mathcal{E}* \mathcal{E}_2)(n) = \mathcal{E}_3(n) = \{0.,0.,1.,4.,14.,40.,105.,256.,594.,1324.,2860.,6020.\}$$

Where [15] is obtained from the 4th derivative of $S_n(x, y, z)$ with x = c0 = 0, y = c1 = 1 and z = c2 = 1. Note that $\mathcal{E} = \mathcal{E}_0$ by definition.

When $S_n(x, y, z) = S_n(1, 1, 0)$ we have the Perrin sequence. Then $\mathcal{E} = Padovan(n)$. The convolution of the Padovan sequence is found in OEIS A104578 the "Padovan Convolution Triangle". The columns of this triangle are successive self-convolutions of the Padovan sequence. Other known element sequences are Fibonacci numbers (A000045), Generalized Gaussian Fibonacci numbers (OEIS A088137), Pell numbers (OEIS A000129), Jacobsthal numbers (OEIS A001045), Tribonacci numbers (OEIS A000073), Mersenne numbers (OEIS A000225) and the Narayana's cow numbers (OEIS A000930). Since every sequence has an element sequence, convolution triangles can be easily constructed from the z derivatives of the ISPs. The derivative of each term n is divided by the moving index n from 2 to n and then all terms are divided by (k-1)! to obtain $\mathcal{E}_{k-1}$.

We close with the following definition and theorem:

*Definition- The **element** sequence $\mathcal{E}(n)$ is obtained from the first derivative of the sequence S_n(**c_0, c_1, c_2**).*

Theorem: The derivatives of the sequence $S_n(c_0, c_1, c_2)$ are convolution operators on the element sequence $\mathcal{E}(n)$.

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Table of the First 12 ISPs $S_n(x,y,z)$

n	ISP for Sequence term $S_n(c_0, c_1, c_2)$ of cubic polynomial $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$
1	z
2	$2y + z^2$
3	$3x + 3yz + z^3$
4	$2y^2 + 4xz + 4yz^2 + z^4$
5	$5y^2z + 5yz^3 + z^5 + 5x(y + z^2)$
6	$3x^2 + 2y^3 + 9y^2z^2 + 6yz^4 + z^6 + 6x(2yz + z^3)$
7	$7x^2z + 7y^3z + 14y^2z^3 + 7yz^5 + z^7 + 7x(y^2 + 3yz^2 + z^4)$
8	$2y^4 + 16y^3z^2 + 20y^2z^4 + 8yz^6 + z^8 + 4x^2(2y + 3z^2) + 8x(3y^2z + 4yz^3 + z^5)$
9	$3x^3 + 9y^4z + 30y^3z^3 + 27y^2z^5 + 9yz^7 + z^9 + 9x^2(3yz + 2z^3) + 9x(y^3 + 6y^2z^2 + 5yz^4 + z^6)$
10	$2y^5 + 10x^3z + 25y^4z^2 + 50y^3z^4 + 35y^2z^6 + 10yz^8 + z^{10} + 5x^2(3y^2 + 12yz^2 + 5z^4) + 10x(4y^3z + 10y^2z^3 + 6yz^5 + z^7)$
11	$11y^5z + 55y^4z^3 + 77y^3z^5 + 44y^2z^7 + 11yz^9 + z^{11} + 11x^3(y + 2z^2) + 11x^2(6y^2z + 10yz^3 + 3z^5) + 11x(y^4 + 10y^3z^2$ $+ 15y^2z^4 + 7yz^6 + z^8)$
12	$3x^4 + 2y^6 + 36y^5z^2 + 105y^4z^4 + 112y^3z^6 + 54y^2z^8 + 12yz^{10} + z^{12} + 8x^3(6yz + 5z^3) + 6x^2(4y^3 + 30y^2z^2 + 30yz^4$ $+ 7z^6) + 12x(5y^4z + 20y^3z^3 + 21y^2z^5 + 8yz^7 + z^9)$
13	$13x^4z + 13y^6z + 91y^5z^3 + 182y^4z^5 + 156y^3z^7 + 65y^2z^9 + 13yz^{11} + z^{13} + 13x^3(2y^2 + 10yz^2 + 5z^4) + 13x^2(10y^3z$ $+ 30y^2z^3 + 21yz^5 + 4z^7) + 13x(y^5 + 15y^4z^2 + 35y^3z^4 + 28y^2z^6 + 9yz^8 + z^{10})$