Homogeneous Linear Differential Equations, Intersequence Polynomials and Binet's Formula

Intersequence polynomials have been found to provide a series solution to general homogeneous linear differential equations. From the last chapter we discussed a solution to a second order differential equation:

> "Consider the (differential) equation y(x)'' – 2y(x)'- y(x) = 0. A solution is y(x) = $(1/a_1)*(e^{\phi 1*x} - e^{\phi 2*x})$ with ϕ1 and ϕ2 solutions to the equation $z*(z^2 - 2z - 1) = 0$ and a_1 a constant. Then the first derivative of the ISP $S_n(0,0,1,2)$ gives the element sequence $\mathcal{E}_0(\mathrm{n})$ $=\{1,2,5,12,29,70,169,408,985,2378 \ldots\}$. It can be shown that a series solution y(x) = $(1/(2\sqrt{2})) * (\sum_{n=1}^{\infty}(\phi 1 * x)^n/n! - \sum_{n=1}^{\infty}(\phi 2 * x)^n/n!)$ is equivalent to the series y(x) = $\sum_{n=1}^{\infty}\mathcal{E}_0(\mathrm{n}) * (x)^n/n!$ and y(x) = $(1/(2\sqrt{2}))*(e^{\phi 1*x} - e^{\phi 2*x})$ where all y(x) satisfy the original differential equation! Let Y[n] = $\sum_{n=1}^{\infty}\mathcal{E}_0(\mathrm{n}) * (x)^n/n!$ be the sequence of coefficients for powers of x. Then the series sequence in x is true for any second order differential equation y(x)'' – Ay(x)'- By(x) = 0."

In this Chapter I further examine these series solutions, how they are obtained and how the method of solution can be extended to higher order differential equations. We are looking at ordinary differential equations of the form,

[1] $$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}) = g(x)$$

This equation is **homogeneous** when g(x) = 0 otherwise it is said to be **non-homogeneous**. Also, the leading coefficients for y and derivatives can be functions of x. These coefficients may be complex or imaginary. Second, third and fourth order equations can be solved using the 4^{th} order intersequence polynomials. Each order is discussed below.

A. Second order homogeneous differential equations.

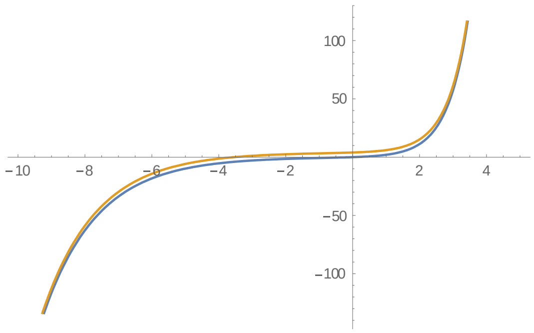
Let y(x)'' – Ay(x)'- By(x) = 0. The fundamental or basis solution set is $\{ e^{\phi 1*x}, e^{\phi 2*x} \}$ where ϕ1 and ϕ2 are the roots of the fundamental equation $z^2 - A z - B = 0$. Use 4^{th} order ISP terms S_n(c0, c1, c2, c3) where c0, c1, c2 and c3 represent the coefficients of $z^4 - c3*z^3 - c2*z^2 - c1*z - c0 = 0$. If we multiply the fundamental equation by z2 and match the representative coefficients we find $z^4 - A*z^3 - B*z^2 = 0$ or $z^4 - c3*z^3 - c2*z^2 = 0$. We then find that all second order ODEs are represented by the ISP, S_n(0, 0, c2, c3). The first derivative of the ISP $S_n(0,0,\mathrm{c2},\mathrm{c3})$ gives the element sequence $\mathcal{E}_0(\mathrm{n})$.

Proposition 1- The equation $\sum_{n=1}^{\infty}\mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ is a solution to the ODE y(x)'' – c3*y(x)'- c2*y(x) = 0.

We can write the solution using the fundamental set as y(x) = $a * e^{\phi 1*x} + b * e^{\phi 2*x}$ where a and b are constants. Binet's formula for the solution can be obtained by using conditions for y(0) and y'(0); y(0) = a + b and y'(0) = a* ϕ1 + b* ϕ2. If the solution requires y(0) = 0 and y'(0) = 1 then we obtain Binet's formula by solving for a, and b. In this case a = 1/ $(\phi 1 - \phi 2)$ and b = -1/ $(\phi 1 - \phi 2)$.

Proposition 2- The equation $\sum_{n=1}^{\infty}\mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ satisfies Binet's formula with y(x) = $(1/(\phi 1 - \phi 2))*(e^{\phi 1*x} - e^{\phi 2*x})$.

Let $S_n(0, 0, 1, 1)$ represent the ISP for the ODE y(x)'' – y(x)'- y(x) = 0. It can be shown that the first derivative gives the Fibonacci sequence
$\mathcal{E}_0(\mathbf{n})$ = $\{1,1,2,3,5,8,13,21,34,55,89,144,233, \ldots\}$. We find that the coefficients in powers of x of the solution $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathrm{n}) * (x)^n/n!$ match the Binet formula solution: y(x) = $0 + 1x + 0.5x^2 + 0.3333x^3 + 0.125x^4 + 0.04166x^5 + \cdots$. A graph of the solutions is shown with **ϕ's** as the golden ratio in Binet's formula.



ISP solution (blue) and Binet formula solution (orange, shifted +4 units) to y(x)'' – y(x)'- y(x) = 0.

B. Third order homogeneous differential equations.

Let y'''(x) - Ay(x)'' – By(x)'- Cy(x) = 0. The fundamental or basis solution set is { $e^{\phi 1 * x}, e^{\phi 2 * x}, e^{\phi 3 * x}$ } where ϕ1, ϕ2 and ϕ3 are the roots of the fundamental equation z^3 -Az^2 – B z - C = 0. Use 4th order ISP terms S_n(c0, c1, c2, c3) where c0, c1, c2 and c3 represent the coefficients of z^4 – c3*z^3- c2*z^2-c1*z - c0 = 0. If we multiply the fundamental equation by z and match the representative coefficients, we find z^4 – A*z^3- B*z^2 – C*z = 0 or z^4 – c3*z^3- c2*z^2 – c1z = 0. We then find that all third order ODEs are represented by the ISP, S_n(0, c1, c2, c3). The first derivative of the ISP $S_n(0, \mathrm{c1}, \mathrm{c2}, \mathrm{c3})$ gives the element sequence $\mathcal{E}_0(\mathrm{n})$.

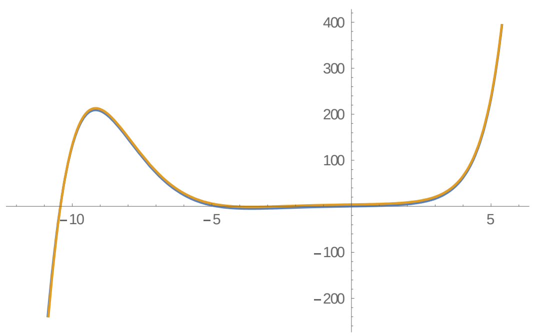
Proposition 3- The equation $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ is a solution to the ODE y(x)''' – c3*y(x)''- c2*y(x)' – c1*y(x) = 0.

We can write the solution using the fundamental set as y(x) = $a * e^{\phi 1 * x} + b * e^{\phi 2 * x} + c * e^{\phi 3 * x}$ where a, b and c are constants. A Binet-like formula for the solution can be obtained by using three conditions for y(0), y(0)' and y''(0);
y(0) = a + b + c , y'(0) = a* ϕ1 + b* ϕ2 + c * ϕ3 and y''(0) = a* $\phi 1^2$ + b* $\phi 2^2$ + c * $\phi 3^2$. If the solution requires y(0) = 0 , y'(0) = 1 and y''(0) = γ then we obtain a Binet-like formula by solving for a, b and c. The condition on the second derivative is determined by $\mathcal{E}_0(\mathbf{n})$.

Proposition 4- The equation $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ satisfies a Binet- like formula using the fundamental set.

Let $S_n(0, 1, 1, 0)$ represent the Perrin type ISP for the ODE y(x)''' – y(x)'- y(x) = 0. It can be shown that the first derivative gives the Padovan sequence
$\mathcal{E}_0(\mathbf{n})$ = $\{1,0,1,1,1,2,2,3,4,5,7,9,12 \ldots\}$. We find that the coefficients in powers of x of the solution $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathrm{n}) * (x)^n/n!$ match the Binet- like formula solution: y(x) = $0 + 1x + 0x^2 + 0.1666x^3 + 0.04166x^4 +$

$0.00833x^5 + 0.00277x^6 + \cdots$. The constant γ is 0. A graph of the solutions is shown with $\boldsymbol{\phi}$'s as the three roots of the Perrin cubic equation $z^3 - z - 1 = 0$ in the Binet-like formula. The constants a, b and c are found to be real and complex. The real constant a is a = $\frac{1}{138}(-2^{1/3}3^{1/6}\sqrt{23}(9-\sqrt{69})^{2/3} + 2^{1/3}3^{1/6}\sqrt{23}(9+\sqrt{69})^{2/3})$ as found by *Mathematica*. As expected, the Binet-like formula can be expressed in radical form but the constants and fundamental sets are quite bulky. The conversion to numerical form with 50 significant figures and $\mathcal{E}_{\mathbf{0}}(\mathbf{n})$ up to powers of x^{40} in the series solution match the analytical solution as shown in the figure below;



ISP solution (blue) and Binet-like formula solution (orange, shifted +4 units) to y(x)''' – y(x)'- y(x) = 0.

C. Fourth order homogeneous differential equations.

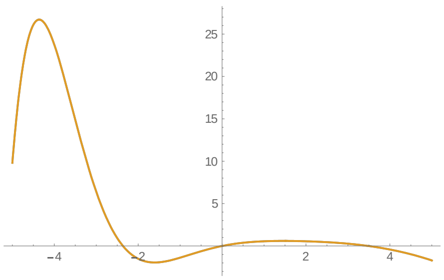
Let y''''(x)- Ay'''(x) - By(x)'' – Cy(x)'- Dy(x) = 0. The fundamental or basis solution set is { $e^{\phi 1 * x}$, $e^{\phi 2 * x}$, $e^{\phi 3 * x}$, $e^{\phi 4 * x}$} where $\phi 1, \phi 2, \phi 3$ and $\phi 4$ are the roots of the fundamental equation $z^4 - Az^3 - Bz^2 - Cz - D = 0$. Use 4th order ISP terms S_n(c0, c1, c2, c3) where c0, c1, c2 and c3 represent the coefficients of $z^4 - c3*z^3 - c2*z^2 - c1*z - c0 = 0$. If we multiply the fundamental equation by 1 and match the representative coefficients, we find $z^4 - A*z^3 - B*z^2 - C*z - D = 0$ or $z^4 - c3*z^3 - c2*z^2 - c1z - c0 = 0$. We then find that all fourth order ODEs are represented by the ISP, S_n(c0, c1, c2, c3). The first derivative of the ISP $S_n(\mathrm{c0, c1, c2, c3})$ gives the element sequence $\mathcal{E}_0(\mathrm{n})$.

Proposition 5- The equation $\sum_{n=1}^{\infty} \mathcal{E}_{\mathbf{0}}(\mathbf{n}) * (x)^n / n!$ is a solution to the ODE y(x)'''' – c3*y(x)'''- c2*y(x)'' – c1*y(x) -c0*y(x) = 0.

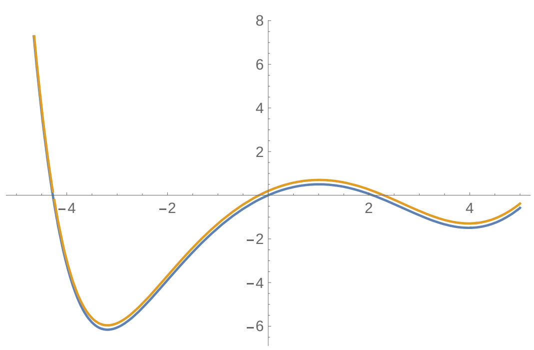
We can write the solution using the fundamental set as y(x) = $a * e^{\phi 1 * x} + b * e^{\phi 2 * x} + c * e^{\phi 3 * x} + d * e^{\phi 4 * x}$ where a, b, c and d are constants. A Binet-like formula for the solution can be obtained by using four conditions for y(0), y(0)', y''(0); and y'''0);

y(0) = a + b + c +d , y'(0) = $a * \phi 1 + b * \phi 2 + c * \phi 3 + d * \phi 4$, y''(0) = $a * \phi 1^2 + b * \phi 2^2 + c * \phi 3^2 + d * \phi 4^2$ and y'''(0) = $a * \phi 1^3 + b * \phi 2^3 + c * \phi 3^3 + d * \phi 3^3$. If the solution requires y(0) = 0 , y'(0) = 1 , y''(0) = γ and y'''(0) = β then we obtain a Binet-like formula by solving for a, b, c and d. The conditions on the second derivative and third is determined by $\mathcal{E}_{\mathbf{0}}(\mathbf{n})$.

Proposition 6- The equation $\sum_{n=1}^{\infty} \mathcal{E}_{\mathbf{0}}(\mathbf{n}) * (x)^n / n!$ satisfies a Binet- like formula using the fundamental set.

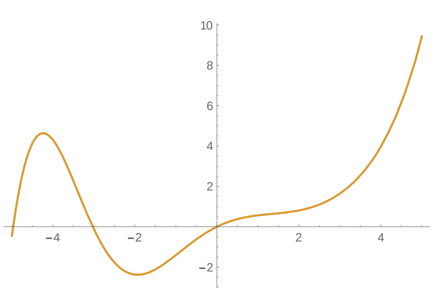


Let $S_n(-1, -1, -1, -1)$ represent the Tetranacci type ISP for the ODE y(x)'''' + y(x)''' + y(x)'' + y(x)' +y(x) = 0. It can be shown that the first derivative gives the inverse if the 5th cyclotomic polynomial (OEIS A010891), $\mathcal{E}_{\mathbf{0}}(\mathbf{n}) = \{1, -1, 0, 0, 0, 1, -1, 0, 0, 0, 1 \ldots\}$. We find that the coefficients in powers of x of the solution $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathrm{n}) * (x)^n/n!$ match the Binet- like formula solution: y(x) = $0 + 1x - 0.5x^2 + 0x^3 + 0x^4 + 0x^5 + 0.001388x^6 - 0.0001984x^7$ …. The constants γ is -1 and β is 0. A good correlation between the second and third element of $\mathcal{E}_{\mathbf{0}}(\mathbf{n})$ and γ and β is found. A graph of the solutions is shown with $\boldsymbol{\phi}$**'s** as the four roots of the tetranacci equation $z^4 + z^3 + z^2 + z + 1 = 0$ in the Binet-like formula. The constants a, b c and d are found to be complex. The real constant a is a = $\frac{(1+(-1)^{1/5})(-2(-1)^{1/5}+2(-1)^{2/5}+(-1)^{4/5})}{5(2-(-1)^{1/5}+2(-1)^{2/5})}$ as found by *Mathematica*. As expected, the Binet-like formula can be expressed in radical form, but the constants and fundamental sets are quite bulky. The conversion to numerical form with 50 significant figures and $\mathcal{E}_{\mathbf{0}}(\mathbf{n})$ up to powers of x^{40} in the series solution match the analytical solution as shown in the figure below;



ISP solution (blue) and Binet-like formula solution (orange, shifted +0.2 units) to y(x)'''' + y(x)''' + y(x)'' + y(x)' + y(x) = 0.

For comparison other solutions to similar fourth order ODEs are shown below.



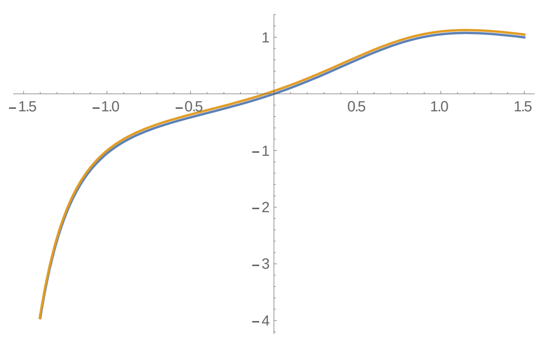
ISP solutions and Binet-like formula solutions to y(x)'''' + y(x)''' + y(x)'' - y(x)' - y(x) = 0 and (above inset) y(x)'''' + y(x)''' + y(x)'' - y(x)' + y(x) = 0.

Proposition 7- The equation $\sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ is a solution to the ODE y(x)'' – c3(x)*y(x)'- c2(x)*y(x) = 0.

We extend the solutions to coefficients which are also functions of the variable x. The most general homogeneous second order equation is

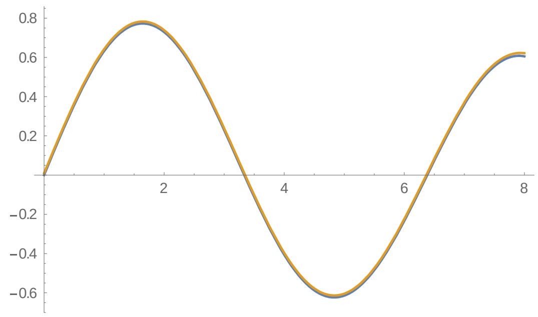
[2] $$a(x)\frac{d^2y(x)}{dx^2} + b(x)\frac{dy(x)}{dx} + c(x)y(x) = 0$$

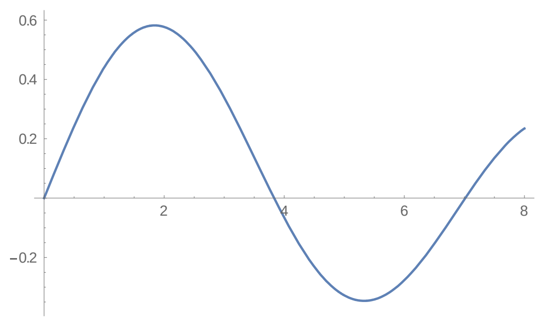
Proposition 7 allows b(x) = c3(x), c(x) = c2(x) and a(x) = 1. The solution is then equivalent to the above section on second order differential equations. The only difference is that the roots to the fundamental equation are functions of x and the constants a and b are also a function of x. As an example, let $S_n(0, 0, 1, 1\text{-}x^2)$ represent the ISP for the ODE $y(x)'' - (1-x^2)*y(x)' - y(x) = 0$. It can be shown that the first derivative gives the sequence $\mathcal{E}_0(\mathbf{n}) = \{1, 1 - x^2, 2 - 2x^2 + x^4, 3 - 5x^2 + 3x^4 - x^6, 5 - 10x^2 + 9x^4 - 4x^6 + x^8, \ldots\}$. We find that the coefficients are themselves powers of x. The roots are also powers of x; $\phi1[x] = \frac{1}{2}\left(1 - x^2 - \sqrt{5 - 2x^2 + x^4}\right)$ *and* $\phi2[x] = \frac{1}{2}\left(1 - x^2 + \sqrt{5 - 2x^2 + x^4}\right)$. The Binet formula constant at x = 0 is $-\frac{1}{\sqrt{5}}$. The series solution in $\mathcal{E}_0(\mathbf{n})$ is corrected by multiplying by $A1 = \frac{\sqrt{5-2x^2+x^4}}{\sqrt{5}}$. A graph of the solutions $A1 * \sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (x)^n/n!$ and $(-\frac{1}{\sqrt{5}})*(e^{\phi1(x)*x} - e^{\phi2(x)*x})$ are shown to be equivalent when graphed.



ISP solution (blue) and Binet formula solution (orange, shifted +0.05 units) to y(x)'' - (1-x²) y(x)' - y(x) = 0.

If we allow a(x) to differ from 1, then our only option using the Binet Formula is to divide all terms by a(x). It seems that this may lead to singularities and using the example of the Bessel equation shows that the singularity is not overcome. Let $S_n(0, 0, -1/x, -(x^2-1)/x^2)$ represent the ISP for Bessel functions of the first kind which solve the ODE $x^2*y(x)'' + x*y(x)' + (x^2-1)\, y(x) = 0$. We have divided the equation by $a(x) = x^2$. Solving this equation using the ISP solution and the Binet formula we find that the first derivative of the ISP is $\mathcal{E}_0(\mathbf{n}) = \left\{1, -\frac{1}{x}, -1 + \frac{2}{x^2}, \frac{-3+2x^2}{x^3}, 1 + \frac{5}{x^4} - \frac{5}{x^2}, \ldots\right\}$. At this point we find the root at x =0 leads to the singularity. Using the roots to the fundamental equation and keeping the Binet coefficients dependent on x, we obtain the graphic solution below.





ISP solution (blue) and Binet formula solution (orange, shifted +0.01 units) to x^2 y(x)'' + x y(x)' + (x^2- 1) y(x) = 0. Below is Mathematica's BesselJ function for n = 1

I find that although the shapes of these curves are the same for all plotted functions, the period and magnitude for the BesselJ function differs from the ISP and Binet Formula. Visually we see a difference in the slope at x = 0 between the curves. The slope is ½ for BesselJ and defined as 1 for the Binet formula. Also, the period suggests that the eigenvalue (roots) differ. However, all functions are found as solutions to x^2*y(x)'' +x*y(x)'+(x^2-1) y(x) = 0 if the coefficient at x = 0 is assumed a constant.

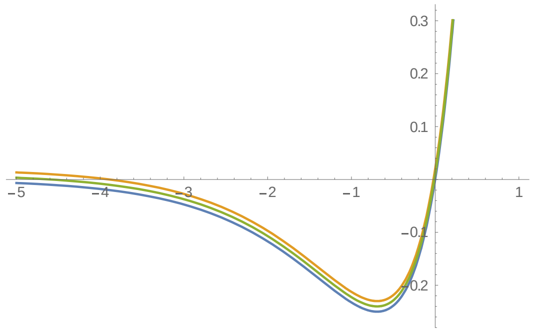
Comparison to the Laplace Transform

The above solutions can be compared to the method of Laplace Transforms for solving second order ODEs. The Laplace transform is a linear operator that reduces differential operators to algebraic equations. It also is an integral convolution operator, integrating a function f(x) with the exponential e^{-sx} over a range of 0 to infinity. The parameter s is a laplace transform variable. The Laplace transform operates on a function f(t) as L[f(t)] = F(s). It also has the property on derivatives that the Laplace transform of the first derivative is L[f'(t)] = s*F[s] – f(0) and the second derivative L[f''(t)] = s^2*F[s] –s f(0) – f'(0). As an example, solve y''(t) -3y'(t) +2y(t) = 0 with the conditions that y(0) = 0 and y'(0) = 1. The Laplace transform leads to the linear algebraic equation (s^2 – 3s +2)*F[s] = 1. The inversion of F[s] provides the solution to the problem. The inversion is performed using the Heavyside expansion theorem,

[3] $$y(t) = L^{-1}[F(s)] = \sum_{j=1}^{n} \frac{p(a_j)}{q'(a_j)} e^{a_j * t}$$

where a_j are the roots of $s^2 - 3s + 2 = 0$ and $\frac{p(s)}{q'(s)} = \frac{1}{2s-3}$.

The solution from [3] is then y(t) = $-1 * \text{Exp}[1 * t] + 1 * \text{Exp}[2 * t]$. This is the same solution obtained using the Binet formula or the ISP $S_n(0, 0, -2, 3)$ with derivative sequence $\mathcal{E}_{\mathbf{0}}(\mathbf{n}) = \{1.,3.,7.,15.,31.,63.,127.,\ldots\}$. The graph below compares the 3 solutions.



ISP solution (blue), Binet formula solution (orange, shifted +0.02 units) and Laplace Transform solution (green, shifted +0.01 units) to y(x)'' - 3y(x)' + 2 y(x) = 0.

The transform agrees in form with the Binet formula. The ISP solution differs by not requiring finding the roots to the fundamental equation. The element sequence provides the coefficients to the exponential forms. As shown above, the Binet solution has some limited application if the constant coefficients obtained from the roots lead to a singularity. The ISP solution also requires a coefficient that does not lead to a singularity.

The Laplace transform shows us a method for solving non-homogeneous ODEs. In the example above, solve the equation y''(x) -3y'(x) +2y(x) = g(x) where g(x) is only a function of x. Then it can be shown using the convolution theorem and the linearity of the Laplace operator that the homogeneous and non-homogeneous solutions are separate and additive. The convolution of the transforms for F(s) and G(s) is

[4] $\quad L[\int_0^t f(\tau)g(t-\tau)\,d\tau] = F(s)*G(s) = L[\int_0^t f(t-\tau)g(t)\,d\tau\,]$

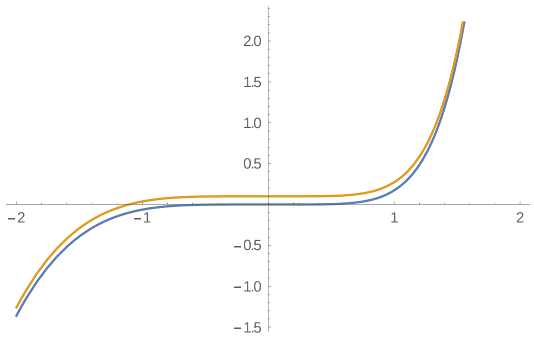
The inverse Laplace transform is then given by the integrals. We notice that the function L^{-1} F(s) is the solution to the homogeneous problem and g(t) is the non-homogeneous term. Using the example from above the solution to the non-homogeneous problem is y(t) = $-1 * \text{Exp}[1 * t] + 1 * \text{Exp}[2 * t] + \int_0^t f(t-\tau)g(t)\,d\tau$

[5] y(t) = $-1 * \text{Exp}[1 * t] + 1 * \text{Exp}[2 * t] + \int_0^t (-1 * \text{Exp}[1 * (t-\tau)] + 1 * \text{Exp}[2 * (t-\tau)]\,)g(t)\,d\tau$

For the Laplace and Binet solutions the integration on the right can be completed manually or using *Mathematica*. If g(t) is a function of a constant and power of x then a simple ISP formula can be used for the integration. Let g(t) = $a*t^j$. Then it can be shown that the integral is expressed by the element $\mathcal{E}_{\mathbf{0}}(\mathbf{n})$ as

[6] $\quad \int_0^t f(t-\tau)g(t)\,d\tau = a * j! * \sum_{\boldsymbol{n}=\mathbf{1}}^{\infty} \mathcal{E}_{\mathbf{0}}(\mathbf{n}) * (\boldsymbol{x})^{(\boldsymbol{n}+\boldsymbol{j}+\mathbf{1})}/(\boldsymbol{n}+\boldsymbol{j}+\mathbf{1})!$

A graph of the non-homogeneous term for g(x) = $2*x^3$ is shown below



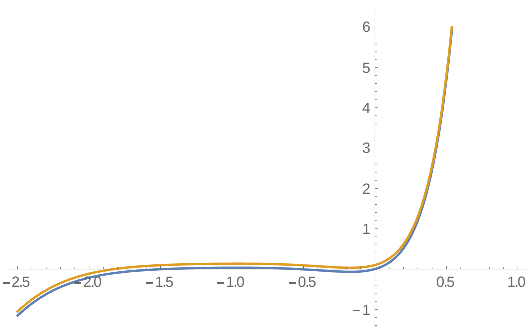
*ISP solution (blue) and Laplace Transform solution (orange, shifted +0.1 units) for the integrated non-homogeneous term for y(x)'' - 3y(t)' + 2 y(t) = 2*t³.* $\int_0^t f(t-\tau)g(t)\,d\tau = \frac{1}{4}(45 + 3e^t(-16 + e^t) + 2t(21 + t(9 + 2t)))$

For higher order non-homogeneous ODEs I find the following pattern for third order [7] and fourth order [8] non-homogeneous solutions,

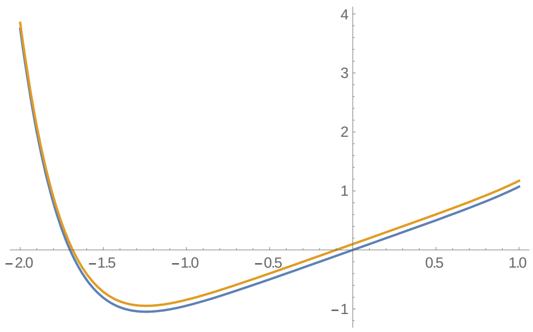
[7] $\int_0^t (\int_0^t f(t-\tau)\,d\tau) g(t)\,d\tau = \int_0^t \left(\int_0^t g(\tau)\,d\tau\right) f(t-\tau)\,d\tau =$
$a * j! * \sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (\boldsymbol{x})^{(n+j+2)}/(\boldsymbol{n}+\boldsymbol{j}+\mathbf{2})!$

[8] $\int_0^t (\int_0^t \int f(t-\tau)\,d\tau\,d\tau) g(t)\,d\tau = \int_0^t \left(\int_0^t \int g(\tau)\,d\tau\,d\tau\right) f(t-\tau)\,d\tau =$

$$a * j! * \sum_{n=1}^{\infty} \mathcal{E}_0(\mathbf{n}) * (\boldsymbol{x})^{(n+j+3)}/(\boldsymbol{n}+\boldsymbol{j}+\mathbf{3})!$$



*ISP solution (blue) and Binet solution (orange, shifted +0.1 units) for the non-homogeneous fourth order ODE; y(x)'''' - 10y(t)''' + 35 y(t)'' - 50y(t)' + 24y(t) = 7*t⁵*



*ISP solution (blue) and Binet solution (orange, shifted +0.1 units) for the non-homogeneous third order ODE: y(x)''' - y(t)'' - y(t) = 7*t*^{5}

In many problems in physics and engineering the non-homogeneous term is an impressed force on an oscillating system. The non-homogeneous term is a sinusoidal function in time give as f_0*sin(α*t). The solution for second order is then a double summation:

[9] $$\sum_{j=0}^{\infty} \sum_{n=1}^{\infty} f_0 * (-1)^j * \alpha^{2j+1} * t^{2j+1} * t^n/(n+2j+1)! * (\mathcal{E}_0(\mathbf{n}))$$

We find that a polynomial solution is always possible for both homogeneous and non-homogeneous ODEs.

The behavior of the polynomial solution to continuous functions was proved in the following theorem:

Theorem (Weierstrass) - Let the function y(x) be continuous on the finite closed interval [a,b]. For any ε >0, there exists a positive integer n and a corresponding polynomial $p_n(x)$ of the nth degree such that |y(x) - $p_n(x)$| < ε for any x € [a,b].

This theorem indicates that an arbitrary continuous function defined on a closed and finite interval can be approximated uniformly by suitable polynomials [1] There exists such a polynomial defined by a combination of coefficients to the set of monomials {x, x^2, x^3, …, x^m}. This set of coefficients is found to be the series $\mathcal{E}_0(\mathbf{n})/\mathbf{n}!$ and forms a fundamental basis of the function y(x). In vector notation these are Fourier coefficients of the basis vectors. The sequence of these coefficients has been found from the derivative of the associated inter sequence polynomials. It will be assumed that solutions for general higher order homogeneous differential equations can be found from the higher order ISPs.

Richard Turk Aug 16, 2019

[1] P. Dennery and A, Krzywicki, Mathematics for Physicists, Harper Row (1967), pg. 200.

Table of ISP First Derivatives of $S_n(w,x,y,z)$

n	First Derivative ISP $(dS_n(q_0, q_1, q_2, q_3)/dz)/n$ for the quartic polynomial $\mathbf{x^4 - q_3 x^3 - q_2 x^2 - q_1 x - q_0 = 0}$
1	1
2	$(2z)/2$
3	$(3y + 3z^2)/3$
4	$(4x + 8yz + 4z^3)/4$
5	$(5w + 5y^2 + 10xz + 5yz^2 + z^4 + z(10yz + 4z^3))/5$
6	$(12wz + 18y^2z + 24yz^3 + 6z^5 + 6x(2y + 3z^2))/6$
7	$(7x^2 + 7y^3 + 42y^2z^2 + 35yz^4 + 7z^6 + 7w(2y + 3z^2) + 7x(6yz + 4z^3))/7$
8	$(24x^2z + 32y^3z + 80y^2z^3 + 48yz^5 + 8z^7 + 8w(2x + 6yz + 4z^3) + 8x(3y^2 + 12yz^2 + 5z^4))/8$
9	$(9w^2 + 9y^4 + 30y^3z^2 + 27y^2z^4 + 9yz^6 + z^8 + 9x^2(3y + 6z^2) + 9w(3y^2 + 4yz^2 + z^4) + 9x(6wz + 12y^2z + 20yz^3 + 6z^5)$ $+ z\big(60y^3z + 108y^2z^3 + 54yz^5 + 8z^7 + 9w(8yz + 4z^3)\big))/9$
10	$(10x^3 + 30w^2z + 50y^4z + 200y^3z^3 + 210y^2z^5 + 80yz^7 + 10z^9 + 5x^2(24yz + 20z^3)$ $+ 10w(6xy + 12y^2z + 12xz^2 + 20yz^3 + 6z^5) + 10x(4y^3 + 30y^2z^2 + 30yz^4 + 7z^6))/10$
11	$(11y^5 + 44x^3z + 165y^4z^2 + 385y^3z^4 + 308y^2z^6 + 99yz^8 + 11z^{10} + 11w^2(3y + 6z^2) + 11x^2(6y^2 + 30yz^2 + 15z^4)$ $+ 11x(20y^3z + 60y^2z^3 + 42yz^5 + 8z^7)$ $+ 11w\big(3x^2 + 4y^3 + 30y^2z^2 + 30yz^4 + 7z^6 + x(24yz + 20z^3)\big))/11$
12	$(72y^5z + 420y^4z^3 + 672y^3z^5 + 432y^2z^7 + 120yz^9 + 12z^{11} + 8x^3(6y + 15z^2) + 6w^2(6x + 24yz + 20z^3)$ $+ 6x^2(60y^2z + 120yz^3 + 42z^5) + 12x(5y^4 + 60y^3z^2 + 105y^2z^4 + 56yz^6 + 9z^8)$ $+ 12w\big(12x^2z + 20y^3z + 60y^2z^3 + 42yz^5 + 8z^7 + 2x(6y^2 + 30yz^2 + 15z^4)\big))/12$