Perrin Numbers, a Hypergeometic Function and Convolution Sequences

Return to a previous expression for calculating the Perrin number. I previously discussed a hypergeometric equation which produced the Perrin number for odd integers[1]. In this chalkboard I will investigate this function and find some interesting results for the convolution of its associated sequences. The hypergeometric equation in *Mathematica* notation is,

[1] $P(n1\ odd, g, h) = \mathrm{n1} * \sum_{i=1}^{\mathrm{imax}+1} (1/(\mathrm{A1}[[i]] + \mathrm{B1}[[i]])) * \mathrm{Hypergeometric2F1}[-\mathrm{A1}[[i]], -\mathrm{B1}[[i]], 1, 1]$

where P(n1 odd) is the Perrin number for an odd integer[2].

Equation [1] contains two integer sequences which are evaluated for each odd integer. These are

[2] $\mathrm{A1} \rightarrow a[i] = a[i-1] - g, with\ a[1] = (\mathrm{n1} - h)/2$

[3] $\mathrm{B1} \rightarrow b[i] = b[i-1] + 2, with\ b[1] = 1$

where for the Perrin number g = h = 3 and i ranges from 1 to imax = $\mathrm{Floor}[(\mathrm{n1} - 3)/6] + 1.$[3]

As an example, when n1 = 23, the sequences are A1 = {10,7,4,1} and B1 = {1,3,5,7}. The summation in equation [1] is then given by 23*((1/11) *11+(1/10) *120+(1/9) *126+(1/8) *8) = 23*28 = 644.

In previous chapters I discuss the convolution of two sequences. Here we have two sequences A1 and B1 which differ for various odd integers and the opportunity to explore their convolution. Recalling the summation convolution of two integer sequences we have,

[4] $\sum_{k=1}^{n} \mathrm{A1}[[k]] * \mathrm{B1}[[n+1-k]]$

Inserting our sequences for n1 = 23 equation [4] gives us the convolution sequence $\{10, 37, 75, 118\}$. These could be coefficients for a polynomial but A1 and B1 are not polynomials. When we test these numbers using an OEIS (OnLine Encyclopedia of Integer Sequences) search we do not find a known sequence. Let's try a larger prime such as n1 = 37. Using equations [2] to [4] we find a longer convolution sequence, $\{17,65,138,230,335,447\}$. Now when we use an OEIS search we do not find a match but a message saying that this sequence appears to be $-x^3+(37/2)*x^2-1/2x$, a polynomial!

What does this mean? We easily find that if x=1,2,3,4,5, or 6, the polynomial produces the values of the convolutions sequence, $17, 65, 138, 230, 335, and\ 447, respectively$. Also, if we substitute 1/37 the polynomial gives $1/37^3$. Given any fraction x = 1/k where k is a prime integer, we find that the denominator is always k^3. When n1 is tested with other odd values, I find that the convolution polynomial is,

[5] $-x^3 + (\mathrm{n1}/2) * x^2 - 1/2 * \mathrm{x}$

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[1] See Chalkboard 15-Calculating the nth term in the Perrin Sequence when n is Prime.

[2] See OEIS A001608, Formula October 14, 2019

[3] Another form of equation [1] is $\sum_{j=0}^{\mathrm{Floor}[(n-g)/(2*g)]} \frac{n*\mathrm{Gamma}[(\frac{1}{2}(n-g*(2j+1))+2j+1)]}{\mathrm{Gamma}[((2j+1)+1)]*\mathrm{Gamma}[(\frac{1}{2}(n-g*(2j+1))+1)]}$

A general polynomial can be developed for any value of g and h in equation [2]. Let g and h be odd integers and g ≥ h. Then for any n1 the convolution sequence is represented by the polynomial,

[6] $$-(g/3) * x^3 + (\mathrm{n1} + (g - h))/2 * x^2 - g/6 * \mathrm{x}$$

As an example, let n1 = 37 and g=7 and h = 5. Then A1 = {16,9,2} and B1 = {1,3,5}. The convolution sequence is {16, 57, 109}. And the polynomial is $-(7/3) * x^3 + (\mathrm{n1} + 2)/2 * x^2 - 7/6 * \mathrm{x}$. The hypergeometric equation results in a value of 2479/3 which indicates it is not a value from an integer sequence. If we test the next prime, n1 = 41 we find that the hypergeometric function results in an integer 1681. After testing other odd integers, I find the following,

Proposition 1- Given two odd integers g and h where g ≥ h, the hypergeometric equation P(n1, g, h) is an integer for n1 = prime(n) – (g-h) and n1|P(n1,g,h)

From this proposition we find that when g = h, the hypergeometric equation is an integer for all primes. In the example above when g-h = 2 the equation is integral for the set of numbers given by *Mathematica*, $\{1,3,5,9,11,15,17,21,27,29,35,39,41,45,51,57,59,65,69,71,77, \dots\}$. The proposition is true for the numbers n1 but does not produce integer values when n1 is not in the set. However, we find exceptions. For example, for g = 5 and h = 3 the two primes 19, and 23 are not in the above set. Equation [1] gives 247/3 and 943/5 and P(n1,5,3)/n1 is 13/3 and 41/5, respectively. When g= 7 and h = 5 equation [1] gives 76/3 and 69 and P(n1,7,5)/n1 is 4/3 and 3, respectively.

Given the two odd integers g and h when g = h we find that the hypergeometric equation expresses numbers from other sequences. The following Table shows 6 sequences whose odd entries for a(n) can be calculated from equation [1].

Values of g and h	OEIS entry	Description	Example a(n1 =37)	Recurrence
9	A007389	7^{th} order maximal independent sets in cycle graph	296	a(n) = a(n-2) + a(n-9)
7	A007388	5^{th} order maximal independent sets in cycle graph	629	a(n) = a(n-2) + a(n-7)
5	A007387	Number of 3^{rd} order maximal independent sets in cycle graph	2590	a(n) = a(n-2) + a(n-5)
3	A001608	Perrin Sequence or Ondrej Such sequence	33004	a(n) = a(n-2) + a(n-3)
1	A099925	Lucas(n) + $(-1)^n$	54018520	a(n) = a(n-2) + a(n-1) + $(-1)^n$
-1	A270363	A binomial sum a(n1-1)	132555063036344727733	

It is apparent that as g = h increases the values for the associated sequence decrease. A change of 17 orders of magnitude are observed as g changes from -1 to 9. Higher order MIS may be anticipated as g increases beyond 9 but these have not been documented in OEIS. The sample number for g = -1 is found in the table by G.C. Greubel and confirmed by the hypergeometric equation. The number of maximal independent sets is a combinatorial number for an n-cycle or an n-gon graph which can be found as the Perrin sequence of numbers. In graph theory an independent set is a set of vertices in a graph in which no two vertices are adjacent, so no edge connects these two vertices. The maximal independent set (MIS) is the largest set of independent sets for a graph. The MIS is not a subset of any independent set. The k-th order MIS is defined in a paper by Yanco and Bagchi.[4] The theory of k-th order MIS is beyond the scope of this chalkboard and the reference and the OEIS entries can serve as a guide to further investigation. However, we find that the higher order equations have recurrence relations that are parallel to the Perrin sequence. The difference is found in the starting terms for these sequences, but the higher numbers a(n) are the sum of the powers of the g roots to the equations $x^g - x^{g-2} - 1 = 0$.

The above proposition 1 concurs with the fact that n1 divides the Perrin number P(n1) when n1 is prime. Although equation [1] is also an integer for odd non-primes n1 when g = h = 3, we find that P(n1)/n1 is not an integer (n1 does not divide P(n1). However, when g and h are equal and odd integers different from 3, the resulting n1|P(n1) if n1 is in the set of primes. This brings up the question of pseudoprimes. For g =h=3 we have the set of pseudoprimes known for the Perrin numbers, the first at 521^2. In the paper by Yanco and Bagchi there are many composites for higher k order MIS numbers. However, these are claimed to have prime factors <k. The k-pseudoprimes are defined if n1|P(n1) and the prime factors are greater than k. No pseudoprimes are found for the 3rd, 5th and 9th order according to reference (2) but the 7th order has one 7-pseudoprime at n1 = 3481 = 51^2. The hypergeometric equation confirms that 3481 is a pseudoprime for g = h = 9 and that n1 divides a number P(3481) that has an order of 10^{226}.

The combinatorial numbers found with the hypergeometric function suggests that there is a similar role played by the convolution sequence. It is a simple calculation to find the convolution sequence using equations [5] or [6] with given values of an odd integer n1, and integers 1 to imax + 1 for the variable x. If we correlate n1 with values of equation [5] obtained from x = imax +1 for g = h = 3, then a search of these numbers on OEIS could potentially show a significant match. Indeed, running through odd integers numbers from two integer sequences are uncovered. The sequences OEIS: A124593 and A117143 are potential candidates. The first sequence is described as the number of 4-indecomposable trees with n nodes. The second as the number of partitions of n in which any two parts differ by at most 3. The later states that an integer is partitioned with no parts greater than 3. As an example a(7) = 13 since we have partitions, [7], [1,1,1,1,1,1,1], [2,1,1,1,1,1], [2,2,1,1,1], [2,2,2,1], [3,1,1,1,1], [3,2,1,1], [3,2,2], [4,3], [5,2], [3,3,1], [4,2,1] and [4,1,1,1]. The partitions [6,1] and [5,1,1] do not belong to this set. Restricted partitions were discussed in a previous chalkboard.[5] For the first sequence a connected graph is 4 decomposable if one can remove some edges and leave a graph with at least two connected components in which every component has at least 4 nodes.

I find that only certain trees are calculated with equation [5]. First the number of nodes, n1 must be odd but n1 translate to the number of trees for n1-3 or n1-4 depending on whether n1 is 5Mod 6 or 1Mod 6. If n1 is 3 Mod 6 then there are no correlating numbers from equation [5] with either of the two

[4] R. Yanco and A Bagchi, kth Order Maximal Independent Sets in Path and Cycle Graphs, Unpublished 1994

[5] See Chalkboard 17 appendix- Some Observations of the Restricted Rogers Ramanujan Identities

sequences. As an example if n1 = 17 = 5 Mod 6 then equation [5] calculates 48 trees which corresponds to a(14 = 17 -3) (tree with 14 nodes in OEIS A124593) and a(13) (48 restricted partitions of 13) in OEIS A117143. The next n1 = 19 = 1 Mod 6 calculates 57 from equation [5] which agrees with a(15 = 19 - 4) (tree with 15 nodes in OEIS A124593) and a(14) (57 restricted partitions of 14) in OEIS A117143. The next number n1 = 21 has no matching entry in either sequence. The pattern is then repeated for n1 = 23 and n1 = 25. This provides a simple method for calculation of both combinatorial numbers;

Proposition 2- The maximum number calculated from the convolution sequence at x = imax + 1, using $-x^3 + (\mathbf{n1}/2) * x^2 - 1/2 * \mathrm{x}$ **where n1 is an odd integer 5 or 1 Mod 6 correlates to the number of 4 indecomposable trees as (n1 – 4) or (n1 – 3) nodes respectively. Also, this number corresponds to the number of restricted partitions of integers (n1- 5) or (n1- 4).**

Looking at just n1 = 3 mod 6, the associated maximal number in the convolution sequence corresponds to less obvious combinatorial sequences found in OEIS A007518 and A051630. In this case all n1 values are multiples of 3 so the maximal number corresponds to entries for n = n1/3 in A007518 or n = n1-2 in A051630. It is still possible that the convolution sequence represents a new combinatoric sequence which has not been reported in OEIS.

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