An Algorithm for higher order Perrin Numbers and the Hypergeometic Function - Appendix

How can we find the complete integer sequence for the maximal independent sets of the n-cycle? In Chapter 39 I presented a hypergeometric function which calculated the sequence numbers for odd integers. This equation reproduced the numbers obtained from the sum of the powers of the g roots to the equations $x^g – x^{g-2} – 1 = 0$, where g is an odd integer. When g = 3 we have Perrin numbers which have been defined as the number of maximal independent sets of an n-cycle of 1st order[1]. We also noted that when g=h, except for pseudoprimes all Perrin numbers of a prime are divisible by the prime number. Higher order MIS equations are defined for odd g > 3. When g = h, based on Proposition 1, all higher order equations of the form $x^g – x^{g-2} – 1 = 0$ have integer sequences in which the nth sequence number is divisible by a prime number n where the resultant is an integer.

Examine equation $x^9 – x^7 – 1 = 0$. *Mathematica* can compute the nine roots to this equation and the sum of powers of the roots can be calculated. If we examine the corresponding OEIS entry A007389, for this sequence we find all odd entries a(n) for n = 3 to 25 are equal to the value of n. We observe that is true for other values of g. This is true for the Perrin sequence for a(3) = 3, a(5) = 5 and a(7) = 7. There appears to be a pattern in the number of odd values of n for which a(n) = n as g increases. For g = 3, 5, 7, and 9 there are 3, 6, 9, and 12 values of a(n) = n respectively. When g= 9 then a(25) = 25 is the last odd number equal to n since a(27) = 36 and a(29) = 58. Here we find another pattern.

Proposition 1A – For integer sequences representing the polynomial $x^g – x^{g-2} – 1 = 0$, we find that there are 2(j)+j odd integers, where j = (g-1)/2 for which a(n) = n at the beginning of the sequence. The (2j + j +2)nd odd integer is a(n) = 2n.

Proposition 2A – Let n be the last odd integer for which a(n) = n for a given g. Then the integer sequence for $x^g – x^{g-2} – 1 = 0$ can be represented by a cubic polynomial based on fitting the numbers a(n), a(n+2), a(n+4), and a(n+6).

Let me show an example. For g = 9 we find the last odd for which a(n) = n is n = 25 (12th odd number greater than 1 based on Proposition 1. Using the Hypergeometric equation we calculate a(25)= 25, a(27)= 36, a(29)= 58, and a(31)= 93. Fitting these to a cubic polynomial (e.g. using the Fit command in Mathematica) I obtain

[1A] $$23.00000 - 1.83333x + 3.50000 + 0.33333x^3 = 23 - \frac{11}{6}x + \frac{7}{2}x^2 + \frac{1}{3}x^3$$

This equation looks surprisingly like equation [6] for the convolution sequence.

[6] $$-(g/3) * x^3 + (\mathrm{n1} + (g - h))/2 * x^2 - g/6 * \mathrm{x}$$

Examining g = 11 and g =13 I find in general the equation

[2A] $$J - \frac{K}{6}x + \frac{g-2}{2}x^2 + \frac{1}{3}x^3$$

Where J is the odd integer preceding the last odd integer for which a(n) = n and K = 11+ (g-9) *3.

[1] R. Bisdorff, J.L. Marichal, *Counting non-isomorphic maximal independent sets of the n-cycle graph*, Journal of Integer Sequences, Vol 11 (2008).

What is the range of predictability of equations [1A] and [2A] with x? Equation [1A] has a domain of x at least {1,2,3,4}. A test of higher values of x, shows that x is an integer and for $1 \le x \le 10$ the correct values of a(n) are calculated based on the Hypergeometric equation and the sum of powers of the roots. In general, we find equation [2A] has an accurate range in a domain of integers of $1 \le x \le g+1$.

For g = 9 what is the number a(24)? Since the sequence can be written as a(n) = a(n-2) + a(n-9), when n is odd then n-9 is even. Since [1A] is accurate to 10 odd integers equal and above 25, it is used to calculate a(n) for n = 25, 27, 29, 31, 33, 35, 37, 39, 41, and 43. Any value for a(n), n even is obtained from a(n) – a(n-2). For a(n – 9) = a(24) = a(33) – a(31) = 143 – 93 = 50 where a(31) and a(33) are calculated from [1A] with x = 4 and 5, respectively. We find that a(34) = a(43) – a(41) = 688 – 533 = 155 where a(41) and a(43) are calculated from [1A] with x = 9 and 10, respectively. Since x = 11 is outside the domain of x, a(45) cannot be calculated from [1A]. The existing data is used to calculate all integer values of a(n); a(24) to a(43). Then a(45) = a(43) + a(36) = 688 + 191 = 879 and a(44) = a(42) + a(35) = 485 + 210 = 695. This leads to the following proposition:

Proposition 3- Equation [2A] calculates g+1 values of a(n) for odd n as x = 1 ,.. (g+1). This set of data is an initial sequence of values of a(n) from which all values of a(n) for even n between min[a(n)] and max[a(n)] can be calculated. From this initiator sequence it is possible to compute the complete sequence of odd and even values representing the equation $x^g - x^{g-2} - 1 = 0$.

Below is a Table of algorithms useful for calculating the value a(n) for odd n for sequences representing $x^g - x^{g-2} - 1 = 0$.

g	Algorithm for a(n)	Minimum n at x = 1	Maximum n at x = g+1
3	$x^3/3 + 1/2\, x^2 + 7/6\, x + 5$	7	13
5	$x^3/3 + 3/2\, x^2 + 1/6\, x + 11$	13	23
7	$x^3/3 + 5/2\, x^2 - 5/6\, x + 17$	19	33
9	$x^3/3 + 7/2\, x^2 - 11/6\, x + 23$	25	43
11	$x^3/3 + 9/2\, x^2 - 17/6\, x + 29$	31	53
13	$x^3/3 + 11/2\, x^2 - 23/6\, x + 35$	37	63
15	$x^3/3 + 13/2\, x^2 - 29/6\, x + 41$	43	73

From this Table it is easy to calculate a(n) for any odd value within the given range. Let g = 15 then 73 is the maximum odd value at x = 15+1 = 16 and a(73) = 2993. This value agrees with the hypergeometric equation and with the sum of the 15 roots taken to the 73rd power. Also, a(42) = a(55) – a(53) = 589(x = 8) – 440(x = 7) = 149.

But wait; finding the entire sequence can be easier! The Table above is only for odd values of n. There is an algorithm for even values of n! Examine the next Table.

Table of algorithms useful for calculating the value a(n) for even n for sequences representing $x^g - x^{g-2} - 1 = 0$.

g	Algorithm for a(n)	Minimum n at x = 1	Maximum n at x = 3+(g-3)/2
3	$x^2 + 2x + 2$	6	10
5	$x^2 + 6x + 7$	12	18

7	$x^2 + 10x + 18$	18	26
9	$x^2 + 14x + 35$	24	34
11	$x^2 + 18x + 58$	30	42
13	$x^2 + 22x + 87$	36	50
15	$x^2 + 26x + 122$	42	58

These algorithms are simple quadratic expressions where the coefficient for x is g-2 and the constant terms are separated by 6k – 1. This algorithm fills in the values of a(n) for even n. Although the domain of x is less than for odd n, it spans enough numbers to be an initiator when combined with the values for odd n. Below is a *Mathematica* program for doing this when g = 9,

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In[1]:= T=Table[x^3/3+(7/2)*x^2-(11/6)*x+23,{x,1,g+1}]

Out[1]= {25,36,58,93,143,210,296,403,533,688}

In[2]:= ST=Table[x^2+14x +35,{x,1,3+(g-3)/2}]

Out[2]= {50,67,86,107,130,155}

In[3]:= AP=Flatten[Table[{ST[[n]],T[[n]]},{n,1,3+(g-3)/2}]]

Out[3]= {50,25,67,36,86,58,107,93,130,143,155,210}

In[4]:= RecurrenceTable[{a[n]==a[n-2]+a[n-9],a[1]==AP[[1]],a[2]==AP[[2]],a[3]==AP[[3]],a[4]==AP[[4]],a[5]==AP[[5]],a[6]==AP[[6]],a[7]==AP[[7]],a[8]==AP[[8]],a[9]==AP[[9]]},a,
 {n,25}]

Out[4]= {50,25,67,36,86,58,107,93,130,143,155,210,191,296,249,403,342,533,485,688,695,879,991,1128,1394}
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These values match 25 values from a(24) to a(48) in OEIS A007389 for the 7th order maximal independent sets (MIS) in a cycle graph.

Are their other integer sequences which can be generated from cubic and quadratic algorithms covering a small range? Mining for these regions may not be as simple as the case of higher order Perrin numbers. These algorithms remain independent of the actual sequence since no information about the sequence except the constant g is given. However, with only this small piece of information a complete integer sequence is generated. The MIS sequences exhibit a varied length of numbers in which a(n) = n (n odd) and a(n) = 2 (n even). This property may be a requirement for finding cubic and quadratic algorithms. Is it possible that this research has implications beyond mathematical sequences? Repeated sequences of nucleotides are found in nucleic acids and these repeats have been implicated in genetic variation of the gene. Can information of repeated sequences be used to obtained algorithms in small regions of DNA or RNA which predict potential variations of gene structure like the near random variation of higher order Perrin sequences?

As an example, I have interpreted the first 153 integers mod 4 in the sequence for g = 23 into nucleotide bases (Cytosine, Guanine, Adenine and Thymine) where I arbitrarily assign 0 = C, 1 = G, 2 = A and 3 = T. The resulting sequence is,

CACACACACACACACACATA
GATAGATAGATAGATAGATA
GGTAGGTAGGTAGGTAGGTA
GGTACGAATGGAAGCAGGTA
CGAATGGGATCAGTTGCGAA
TGGGATCAGTAGGGTAAGTG
CTAATTCGTGGACCGGACCAG
GAGGTTAAATGCA

This sequence of bases begins with the CAC pattern and then another pattern ATAG emerges. Another pattern AGGT follows, after which the pattern appears to become random. This apparent randomness however is described by powers of the roots of the equation $x^{23} – x^{21} – 1 = 0$!

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