Discriminants and the Element Sequence

I return in this Chapter to Inter sequence polynomials (ISP) of a general cubic monic polynomial derived from the cubic equation $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$ and previously expressed as linear recurrences,

[1] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = S_n(x,y,z)$

These polynomials are known to express powers of n,

[2] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}) = \psi^n + \psi_1^n + \psi_2^n$

where n = 0, 1, 2, …. N and ψ_i are solutions to the cubic equation and $\mathbf{x = c_0}$, $\mathbf{y = c_1}$, $\mathbf{z = c_2}$. The integer sequence associated with $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ is called the **parent** sequence.

The derivative of the ISP ***$S_n(1,1,0)$*** which represent the Perrin sequence, have the following derivative relationship

[3] $\partial_z S_n(1,1,0) = (n) * Pd(n),$

where Pd(n) is the Padovan sequence (OEIS A000931). It was shown that these relationships are generalized for any sequence obtained from a cubic equation $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0.}$ The **element** sequence, $\mathcal{E}$(n) is obtained from the first derivatives of any cubic ISP $S_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$,

[4] $\partial_z S_n(\mathrm{x,y,z}) = n * \mathcal{E}(\mathrm{n})$

In this Chapter, I examine the following proposition.

Proposition 1 – There exists a proportional relationship between some cubic ISP S_n(x, y, z) and its element $\mathcal{E}$(n) sequence of the form $\mathcal{E}$(n) = round(r* S_n(x, y, z)) where r is a constant.

It is important to note that the element sequence cannot be described itself by an ISP. The element sequence can only be obtained by the first derivative of an ISP, or a generating function and in some cases by Proposition 1. Furthermore, we can propose a form for the constant r.

Proposition 2- There exists a constant r for some ISP which is calculated from the root of the cubic equation $c_0*x^3 + fc*x^2 + D = 0$. Here, fc is an integer, D is the discriminant of the cubic equation for $S_n(c_0, c_1, c_2)$; $x^3 - c_2 x^2 - c_1 x - c_0$ and c_0 is a constant (usually an integer for integer sequences).

Proposition 3- The constant r is the limiting ratio of the nth element of the corresponding sequences for element and parent S_n(x, y, z).

These last two propositions suggest that the constant fc is obtained from the equation $\mathbf{c_0*x^3 + fc*x^2 + D = 0}$ where x approaches the inverse of the limiting ratio of the element over the parent sequence (1/r). If we hypothesize that fc is an integer, then as we increase the precision of the ratio x by using higher powers of n (see equation [2]), then fc will approach this integer. In most cases the accuracy of r is in proportion to n so one can use a smaller value of n as an approximation for x. In a previous Chapter the polynomials for $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2})$ and $\partial_z S_n(\mathrm{x,y,z})$ are given for values of n up to 13. Without knowing the sequence we can approximate this ratio for any cubic equation from the 13th ISP, S_{13}(x,y,z) as (ra):

[5] ra =

$$\frac{13x^4+130x^2y^3+13y^6+260x^3yz+390xy^4z+1170x^2y^2z^2+273y^5z^2+260x^3z^3+1820xy^3z^3+1365x^2yz^4+910y^4z^4+2184xy^2z^5+364x^2z^6+1092y^3z^6+936xyz^7+585y^2z^8+130xz^9+143yz^{10}+13z^{12}}{(13(26x^3y^2+13xy^5+13x^4z+130x^2y^3z+13y^6z+130x^3yz^2+195xy^4z^2+390x^2y^2z^3+91y^5z^3+65x^3z^4+455xy^3z^4+273x^2yz^5+182y^4z^5+364xy^2z^6+52x^2z^7+156y^3z^7+117xyz^8+65y^2z^9+13xz^{10}+13yz^{11}+z^{13}))}$$

This equation can be easily programed to calculate ra for any value of x = c_0, y= c_1 and z = c_2 in S_{13}(x,y,z).

The constant integer fc is then given by the following;

[6] $$fc = \text{round}((-D) - c_0 * (1/\text{ra}))^3/(1/\text{ra})^2)$$

Mathematica can be used to find the discriminant of **$x^3 - c_2 x^2 - c_1 x - c_0$** and the root of **$c_0 * x^3 + fc * x^2 + D = 0$.** The constant **r** is the inverse of this root!

I have checked all class 3 negative discriminants (D = {-23, -31, -59, -83, -107, -139, -211, -283, -307, -331, -379, -499, -547, -643, -883, -907}) which I mentioned in a previous chapter. Proposition 1 is true for all these sequences. I have tabulated the results below,

Table of Class 3 Discriminants and the element and parent sequences where $\mathcal{E}$(n) = round(r* S_n(x, y, z)) ,r is a constant.

Discriminant	$S_n(c_0, c_1, c_2)$*	ra	fc	r (20 significant figures)
-23	S_n(1,1,0)	0.3076923	-1	0.31062882964046707776
-31	S_n(1,0,1)	0.41666666	3	0.41723798792621877762
-59	S_n(1,0,2)	0.38215972	6	0.38215952590601216354
-83	S_n(1,2,2)	0.26405347	2	0.26405347425812729156
-107	S_n(2,3,1)	0.23040749	-3	0.23040632820823439562
-139	S_n(2,5,3)	0.17651131	-7	0.176511318642790785
-211	S_n(2,-1,3)	0.33052436	17	0.33052442863713471776
-283	S_n(1,0,4)	0.2391234	12	0.23912340710533188516
-2763 = 9*-307	S_n(4,1,5)	0.17247415	59	0.1724741516059770661
-331	S_n(1,2,4)	0.19847218	8	0.19847218167051375023
-1516 = 4*-379	S_n(2,5,6)	0.13148848	11	0.13148847979736964207

-1996 = 4*-499	$S_n(4,5,4)$	0.15608431	23	0.15608431134369599711
-547	$S_n(3,-2,4)$	0.27950486	32	0.27950486912929793426
-643	$S_n(1,6,10)$	0.089594193	-6	0.089594192478119491621
-883	$S_n(2,5,5)$	0.14564901	5	0.14564901752495315765
-907	$S_n(2,-1,5)$	0.20652066	29	0.20652066223459550 94

*This is the reverse notation of "signature" used by OEIS

As an example, we can consider D = -31 where the element sequence is known as the Narayana's cows sequence (see OEIS A000930) and the parent sequence is OEIS A001609. Calculating r using the root to 50 significant figures and looking at the Tables for n = 316 ([1]),([2]) we find that

r = 0.41723798792621877762147551641029254710239234348115 16

A001609(316) = 4209177984747855163188144169865512269249444645171 9015

A000930(316) = 1756228953179531478680561900055738487863083661850 8981

These numbers agree with Proposition 1 and values of A000930(n) are exact using all values of A001609(n) and Proposition 1. It is noted that for the lower values of discriminant the first 2-4 numbers in a sequence may not be exact. By the 5th number in a sequence the rounding error is reduced, and the correct integer value is displayed. Although some sequences follow the above propositions, not all element sequences can be calculated from the parent by these rules. For example, for D = -23, the sequence $\mathcal{E}(n)$ = round (r* $S_n(1,1,0)$) is the Padovan sequence and the statement is true. The statement is not true for element sequences of $S_n(2,1,0)$, $S_n(3,1,0)$ or of any parent $S_n(\mathbf{c_0}, \mathbf{c_1}, 0)$ where $c_0 > c_1$. The Proposition 1 is also not true for $S_n(c_0, 0, c_2)$ when $c_0 > c_2$ but in a specific case $S_n(c_0,5,6)$ is only true when $1 < c_0 < 7$.

The truth of the proposition does not appear to be dependent on the discriminant since in the later example the discriminant of the cubic equation for $S_n(7,5,6)$ is -9751 = 49*(-199) and the proposition is true. There is some evidence that the proposition is true for class numbers divisible by 3 but this is not always the case. For example, $S_n(4,5,6)$ represents a cubic equation of class number 12 (based on *Mathematica*) and $S_n(3,0,1)$ is also of class number 12 but the former is true and later false to Proposition 1. However, the cubic equation for $S_n(4,5,6)$ has a discriminant of -4648 = 56*-83 where -83 is a class number 3 as shown in the Table above.

A check of various values of ($\mathbf{c_0}$, $\mathbf{c_1}$, $\mathbf{c_2}$) suggest a pattern of "true" or "false" for Proposition 1. One hypothesis indicates "true" when $\mathbf{c_0} + \mathbf{c_2} \geq \mathbf{c_1}$. Another suggests "false" if $\mathbf{c_0} > \mathbf{c_2}$ and $\mathbf{c_0} > \mathbf{c_1}$ with $\mathbf{c_0}$ larger or equal to some integer. At this time, there is insufficient data to determine the exact conditions required for a sequence to be true or false to Proposition 1. A *Mathematica* program is available to

[1] Harvey P. Dale and T. D. Noe, Table of n, a(n) for n = 0..5000 [The first 500 terms were computed by T. D. Noe] in OEIS A000930.
[2] Indranil Ghosh, Table of n, a(n) for n = 1..6012 (terms 1..500 from T. D. Noe) in OEIS A001609.

determine whether the parent sequence for the equation $\mathbf{x^3 - c_2 x^2 - c_1 x - c_0 = 0}$ and its derivative element sequence are related by proposition 1.

When considering second order polynomials $S_n(0,c_1,c_2)$, where the cubic polynomial is $\mathbf{(x^2 - c_2 x - c_1)x = 0}$, the constant r is readily given by the discriminant as r = $\frac{\sqrt{D}*|c_1|}{D}$ and by equation [6]. In general, for second order parent and element sequences (Lucas and Fibonacci types) the proposition is true when $\mathbf{|c_2| \geq |c_1|}$. For example, the Lucas parent sequence $S_n(0,1,1)$ (OEIS A000032) and its Fibonacci sequence $\mathcal{E}(n)$ (OEIS A000045) are related by Proposition 1 as;

[7] $$\mathcal{E}(n) = \text{round}(\frac{\sqrt{5}*1|}{5} * S_n(0, 1, 1))$$

I find for example $\mathcal{E}(38)$ = A000045(38) = 39088169 = round($\frac{\sqrt{5}*1|}{5}$* A000032(38)) = round(0.447213595.. *87403803) = 39088169.

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