Inter Sequence Polynomials ,Linear Recurrent Sequences and Bell Polynomials

Inter sequence polynomials (ISPs) of a general quartic polynomial derived from the fourth order equation $\mathbf{x^4 - c_3 x^3 - c_2 x^2 - c_1 x - c_0 = 0}$ have previously been expressed as linear recurrences,

[1] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}) = S_n(w,x,y,z)$

These polynomials are known to express each term in powers of n,

[2] $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}) = \psi_1^n + \psi_2^n + \psi_3^n + \psi_4^n$

where n = 0, 1, 2, …. and ψ_i are solutions to the quartic equation and $\mathbf{w = c_0}$, $\mathbf{x = c_1}$, $\mathbf{y = c_2}$, $\mathbf{z = c_3}$. The integer sequence associated with $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3})$ is called the **parent** sequence.

Since ISPs are functions of the variables w, x, y and z the z derivative of the ISP $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3})$ has been labeled as the **element** sequence. This element sequence, $\mathcal{E}_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3})$ is obtained from the first derivatives of any quartic ISP $S_n(\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3})$,

[3] $\partial_z S_n(\mathrm{w,x,y,z}) = n * \mathcal{E}_n(\mathrm{w,x,y,z})$

The first four ISPs are the initial sequence terms for the parent and element sequences and are shown in the following table.

nth Term	Parent S_n(w,x,y,z)	Element $\partial_z S_n(\mathrm{w,x,y,z})/n$
1	z	1
2	$2y + z^2$	z
3	$3x + 3yz + z^3$	$y + z^2$
4	$4w + 2y^2 + 4xz + 4yz^2 + z^4$	$x + 2yz + z^3$

Proposition 1 – The first four terms n = 1,2,3,4 of any Parent sequence of fourth order based on inter sequence polynomials is obtained from the coefficients c_i of $\xi^4 - c_3 \xi^3 - c_2 \xi^2 - c_1 \xi - c_0$.

As previously stated, the element sequence cannot be described by an ISP. The element sequence can only be obtained by the first derivative of an ISP, or a generating function.

A linear recurrent sequence (LRS) is defined as a sequence of numbers U_0, U_1, U_2, … and the coefficients c_i where i = 0, 1, 2, 3….(see footnote).[1] For a fourth order recurrent sequence the c_i are given as above by $\mathbf{c_0}$, $\mathbf{c_1}$, $\mathbf{c_2}$, $\mathbf{c_3}$. A fourth order LRS is expressed as,

[4] $U_n = c_0 * U_{(n-4)} + c_1 * U_{(n-3)} + c_2 * U_{(n-2)} + c_3 * U_{(n-1)}$

The major difference between an ISP sequence and an LRS is the nature of the initial sequence terms (U_0, U_1, U_2, U_3). The LRS is not limited to the first terms defined by the ISP. However, the LRS and ISP sequences are based on the same eigenvalues. The general fourth order LRS is also expressed with the same eigenvalues as in [1] but each term is multiplied by a leading coefficient. This coefficient is dependent of the **multiplicity** of the eigenvalues. If $\mathbf{\xi^4 - c_3 \xi^3 - c_2 \xi^2 - c_1 \xi - c_0 = 0}$ has t = r= 4 distinct eigenvalues (roots), then

[1] See Tony Forbes Linear Recurrence Sequences (Talks given by LSBU) November 2014. www.maths.qmul.ac.uk

[5] $U_n = q_1 * \psi_1^n + q_2 * \psi_2^n + q_3 * \psi_3^n + q_4 * \psi_4^n$

where q_i are real or complex numbers. When ψ_i occurs with multiplicity t_i with t distinct values and assuming a fourth order sequence, then $r = 4 = \sum_{j=1}^{t} t_j$. I will show below that the general coefficients q for any multiplicity (of order 4) is,

[6] $$U_n = \sum_{i=1}^{t} \sum_{j=0}^{t_{i-1}} q_{i,j} * n^j * \psi_i^n$$

where n = 0, 1, 2, …

There are several methods for finding the eigenvalues of a general 4th order equation. Since all equations less than 5th order have radical solutions it is possible to use programs such as *Mathematica* to solve for roots of the characteristic polynomial, $\xi^4 - c_3 \xi^3 - c_2 \xi^2 - c_1 \xi - c_0 = 0$. Unfortunately, these algorithms result in complex infinities when the multiplicity of the root is either 3 or 4. I find that for certain values of $c_0 = w$, $c_1 = x$, $c_2 = y$ and $c_3 = z$ the following denominator required to calculate a root is zero;

$$(27x^2 - 72wy - 2y^3 + 9xyz - 27wz^2 + \sqrt{-4(-12w + y^2 - 3xz)^3 + (27x^2 - 72wy - 2y^3 + 9xyz - 27wz^2)^2}$$

when multiplicity $t_i = 3$ or 4.

Two alternate methods for finding the root are 1) matrix methods and 2) use of equation [1] with the first 4 ISPs. (in general, any characteristic equation of order m can be solved by either method).

1) Using matrix methods, the 4x4 matrix is written as $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w & x & y & z \end{bmatrix}$ or alternatively, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_0 & c_1 & c_2 & c_3 \end{bmatrix}$ and can be diagonalized. The diagonal elements are the roots to the characteristic polynomial.
2) Using ISPs, a system of simultaneous equations can be solved by various methods. In *Mathematica* the 'Solve[]' command is used. An example is given for any $S_n(w,x,y,z)$

[7] $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = z$
$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 2y + z^2$
$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 = 3x + 3yz + z^3$
$\lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^4 = 4w + 2y^2 + 4xz + 4yz^2 + z^4$

These equations are solved simultaneously and produce the four eigenvalues (λ_i) exhibiting any multiplicity. Note we are using the parent sequence initial values for the determination of these eigenvalues and all coefficients $q_i = 1$ in [5].

Next use these eigenvalues to find the coefficients to equation [6]. There are five conditions:

1. All eigenvalues are distinct (degenerate condition). Let (U_0, U_1, U_2, U_3) be the desired initial conditions of the sequence. For example (0, 0, 0, 1) or (1, 1, 1, 1). Then solve the following to obtain q_i (i = 1,2,3,4);

[8] $q_1 * \lambda_1 + q_2 * \lambda_2 + q_3 * \lambda_3 + q_4 * \lambda_4 = U_0$

$q_1*\lambda_1^2 + q_2*\lambda_2^2 + q_3*\lambda_3^2 + q_4*\lambda_4^2 = U_1$
$q_1*\lambda_1^3 + q_2*\lambda_2^3 + q_3*\lambda_3^3 + q_4*\lambda_4^3 = U_2$
$q_1*\lambda_1^4 + q_2*\lambda_2^4 + q_3*\lambda_3^4 + q_4*\lambda_4^4 = U_3$

The coefficients are used to find U_n;

[9] $U_n = q_1*\lambda_1^n + q_2*\lambda_2^n + q_3*\lambda_3^n + q_4*\lambda_4^n$

Proposition 2 – If (0, 0, 0, 1) is the initial sequence of a fourth order sequence, the LRS in [9] is equal to the element sequence of the parent sequence for $\xi^4 - c_3\,\xi^3 - c_2\,\xi^2 - c_1\,\xi - c_0$ obtained using inter sequence polynomials. Then $U_n = \mathcal{E}_n(c_0, c_1, c_2, c_3)$.

This proposition is also true for all the conditions found below.

2. Two sets of eigenvalues (λ_1, λ_1), (λ_2, λ_2) are distinct (non-degenerate condition). Let (U_0, U_1, U_2, U_3) be the desired initial conditions of the sequence. Then solve the following to obtain q_i (i = 1,2,3,4);

[10] $q_1*\lambda_1 + q_2*\lambda_1 + q_3*\lambda_2 + q_4*\lambda_2 = U_0$
$q_1*\lambda_1^2 + 2q_2*\lambda_1^2 + q_3*\lambda_2^2 + 2q_4*\lambda_2^2 = U_1$
$q_1*\lambda_1^3 + 3q_2*\lambda_1^3 + q_3*\lambda_2^3 + 3q_4*\lambda_2^3 = U_2$
$q_1*\lambda_1^4 + 4q_2*\lambda_1^4 + q_3*\lambda_2^4 + 4q_4*\lambda_2^4 = U_3$

The coefficients are used to find U_n;

[11] $U_n = q_1*\lambda_1^n + n*q_2*\lambda_1^n + q_3*\lambda_2^n + n*q_4*\lambda_2^n$

3. One set of eigenvalues (λ_1, λ_1) and 2 other (λ_2, λ_3) distinct (non-degenerate condition). Let (U_0, U_1, U_2, U_3) be the desired initial conditions of the sequence. Then solve the following to obtain q_i (i = 1,2,3,4);

[12] $q_1*\lambda_1 + q_2*\lambda_1 + q_3*\lambda_2 + q_4*\lambda_3 = U_0$
$q_1*\lambda_1^2 + 2q_2*\lambda_1^2 + q_3*\lambda_2^2 + q_4*\lambda_3^2 = U_1$
$q_1*\lambda_1^3 + 3q_2*\lambda_1^3 + q_3*\lambda_2^3 + q_4*\lambda_3^3 = U_2$
$q_1*\lambda_1^4 + 4q_2*\lambda_1^4 + q_3*\lambda_2^4 + q_4*\lambda_3^4 = U_3$

The coefficients are used to find U_n;

[13] $U_n = q_1*\lambda_1^n + n*q_2*\lambda_1^n + q_3*\lambda_2^n + q_4*\lambda_3^n$

4. One set of eigenvalues $(\lambda_1, \lambda_1, \lambda_1)$ and 1 other (λ_2) distinct (non-degenerate condition). Let (U_0, U_1, U_2, U_3) be the desired initial conditions of the sequence. Then solve the following to obtain q_i (i = 1,2,3,4);

[14] $q_1*\lambda_1 + q_2*\lambda_1 + q_3*\lambda_1 + q_4*\lambda_2 = U_0$
$q_1*\lambda_1^2 + 2q_2*\lambda_1^2 + 4q_3*\lambda_1^2 + q_4*\lambda_2^2 = U_1$
$q_1*\lambda_1^3 + 3q_2*\lambda_1^3 + 9q_3*\lambda_1^3 + q_4*\lambda_2^3 = U_2$
$q_1*\lambda_1^4 + 4q_2*\lambda_1^4 + 16q_3*\lambda_1^4 + q_4*\lambda_2^4 = U_3$

The coefficients are used to find U_n;

[15] $U_n = q_1*\lambda_1^n + n*q_2*\lambda_1^n + n^2*q_3*\lambda_1^n + q_4*\lambda_2^n$

5. One set of eigenvalues ($\lambda_1, \lambda_1, \lambda_1, \lambda_1$) (non-degenerate condition). Let (U_0, U_1, U_2, U_3) be the desired initial conditions of the sequence. Then solve the following to obtain q_i (i = 1,2,3,4);

[16] $q_1*\lambda_1 + q_2*\lambda_1 + q_3*\lambda_1 + q_4*\lambda_1 = U_0$
$q_1*\lambda_1^2 + 2q_2*\lambda_1^2 + 4q_3*\lambda_1^2 + 8q_4*\lambda_1^2 = U_1$
$q_1*\lambda_1^3 + 3q_2*\lambda_1^3 + 9q_3*\lambda_1^3 + 27q_4*\lambda_1^3 = U_2$
$q_1*\lambda_1^4 + 4q_2*\lambda_1^4 + 16q_3*\lambda_1^4 + 64q_4*\lambda_1^4 = U_3$

The coefficients are used to find U_n;

[17] $U_n = q_1*\lambda_1^n + n*q_2*\lambda_1^n + n^2*q_3*\lambda_1^n + n^3*q_4*\lambda_1^n$

As an example of condition 5, let ξ^4 - **12** ξ^3 **+ 54** ξ^2 **– 108** ξ **+ 81** = 0. The inter sequence polynomial is S_n(**c_0**, **c_1**, **c_2**, **c_3**) = S_n(-81,108,-54,12). From [7] the eigenvalues are $\lambda_1= \lambda_2= \lambda_3= \lambda_4=3$. Let the initial sequence be (0, 0, 0, 1) then from [16] I find q1= -1/81, q2 = 11/486, q3 = -1/81 and q4 = 1/486. From [17] the sequence is U_n = {0,0,0,1,12,90,540,2835,13608, 61236, … This can be simplified to a single equation

[18] $U_n = \frac{1}{2}3^{-5+n}(-6 + 11n - 6n^2 + n^3)$

This sequence is in agreement with the element sequence of S_n(-81,108,-54,12). The advantage of LRS is in programing sequences in convenient notation as [18] shows. Comparison of the LRS and ISP shows the sequence shift as $U(ISP)_n = U(LRS)_{n+3}$.

The form of [18] can be used to find the location of zeros in a sequence. For example, if we choose the initial conditions (U_0, U_1, U_2, U_3) = (-2, -9, -13, 9) in the example above one obtains the following single equation:

[19] $U_n = \frac{1}{2}3^{-3+n}(70 - 164n + 65n^2 - 7n^3)$

Equating the term in parenthesis to zero and solving for n we find n = 5 indicating that $U_5 = 0$. Given an LRS over the integers is $U_n \neq 0$ for all $n \geq 0$? This problem is called Skolem's problem and proposed in 1934 it remains an open problem. The situation is decidable for order 3 but it is not known if it is decidable for other orders. (see footnote)[2].

Bell Polynomials

The ISPs and their derivatives are closely tied to the Bell Polynomials. These polynomials were described by Eric Bell in 1934 to describe partitions of sets. They are multivariate and can be used to calculate all the Inter sequence polynomials and their derivatives. A derivation of the following formula can be found in Birmajer [3].

[2] See V. Blondel and N. Portier, "The presence of a zero in an integer linear recurrent sequence is NP-hard to decide", Linear Algebra and its Applications 351-352 (2002) pp 91-98.
[3] See D. Birmajer, J.B. Gil, and M.D. Weiner, "Linear recurrence Sequences and their Convolutions via Bell Polynomials" arXiv:1405.7727v2, 26 Nov 2014.

The Bell Polynomial can be used to describe ISPs of any order. I will show formula for 4th order, following the analysis in the previous section. For calculation purposes, the partial Bell polynomial described in the literature is written as $Y_{n,k}(x_1,\ldots,x_{n-k+1})$ and can be calculated in Mathematica by the command {BellY[n,k,{x_1,…,x_{n-k+1}}]].

A. The parent sequence of any 4th order polynomial ξ^4 - $\mathbf{c_3}\ \xi^3 - \mathbf{c_2}\ \xi^2 - \mathbf{c_1}\ \xi - \mathbf{c_0} = \xi^4$ - $\mathbf{z}\ \xi^3 - \mathbf{y}\ \xi^2 - \mathbf{x}\ \xi - \mathbf{w}$ is expressed by the Bell polynomial as,

   [20] S_n(w,x,y,z) = $\sum_{k=1}^{n}(k-1)!/(n-1)! * \mathrm{Y}[n,k,\{1!*z,2!*y,3!*x,4!*w\}]$

   The first four polynomials of [20[are $\{z,2y+z^2,3x+3yz+z^3,4w+2y^2+4xz+4yz^2+z^4\}$ which agree with the first four ISPs.

B. The element sequence of any 4th order polynomial ξ^4 - $\mathbf{c_3}\ \xi^3 - \mathbf{c_2}\ \xi^2 - \mathbf{c_1}\ \xi - \mathbf{c_0} = \xi^4$ - $\mathbf{z}\ \xi^3 - \mathbf{y}\ \xi^2 - \mathbf{x}\ \xi - \mathbf{w}$ is expressed by the Bell polynomial as,

   [21] $\mathcal{E}_n$(w,x,y,z) = $\sum_{k=0}^{n-2}(k-2)!/(n-2)! * \mathrm{Y}[n-2,k,\{1!*z,2!*y,3!*x,4!*w\}]$

   The first four non-zero polynomials of [21[are $\{1,z,y+z^2,x+2yz+z^3\}$ which agree with the first element polynomials.

C. The mth derivative of the Parent ISP sequence of any 4th order polynomial ξ^4 - $\mathbf{c_3}\ \xi^3 - \mathbf{c_2}\ \xi^2 - \mathbf{c_1}\ \xi - \mathbf{c_0} = \xi^4$ - $\mathbf{z}\ \xi^3 - \mathbf{y}\ \xi^2 - \mathbf{x}\ \xi - \mathbf{w}$ is expressed by the Bell polynomial as,

   [22] $\partial^m{}_z \boldsymbol{S_n}$(w,x,y,z) = $\sum_{k=0}^{n-(m+1)}\binom{k+m-1}{k} * k!/(n-(m+1))! * \mathrm{Y}[n-(m+1),k,\{1!*\mathrm{z},2!*\mathrm{y},3!*\mathrm{x},4!*\mathrm{w}\}]$

   These derivatives represent the mth convolution of sequences!

I had originally proposed ISP's as a new independent way of looking at integer and many linear sequences without using eigenvalues; showing that they equal the powers of the sum of the eigenvalues of a characteristic polynomial. As I discovered many applications of linear recurrences, I also researched other past work in this field. It was only a few days before writing this chapter that I discovered the interesting applications of Bell Polynomials and how it is directly connected to inter sequence polynomials. I hope readers of these chapters appreciate how mathematical concepts apply in many fields of study and with perseverance can eventually be brought together in a new understanding.

R. Turk January 3, 2020