Bell Polynomials and Perrin and Padovan Sequences

Many of the previous Chapters have been devoted to the Perrin sequence and general integer sequences. In this Chapter I will show that the Bell Polynomial introduced in the last Chapter of third order expresses sequences such as the Perrin and Padovan sequences. I showed that the Perrin numbers can be calculated using various closed expressions such as the hypergeometric, incomplete Beta functions and by the sum of powers of roots to its characteristic equation. Unlike these equations, the modified Bell polynomial is generalized to linear integer recurrences of which the Perrin sequence is a third order subset. As in previous Chapters, I describe the Perrin sequence as the parent sequence to the element Padovan sequence. Element sequences are the first convolution sequences of Parent sequences. I showed that the first derivative of Inter Sequence Polynomials is equivalent to derivatives of the modified Bell Polynomial.

For calculation purposes, the partial Bell polynomial described in the literature is written as $Y_{n,k}(x_1,\ldots,x_{n-k+1})$ and can be calculated in *Mathematica* by the command {BellY[n,k,{x_1,…,x_{n-k+1}}]. The following summarizes 3rd order sequences:

The parent sequence of any 3rd order polynomial $\xi^3 - c_2\xi^2 - c_1\xi - c_0 = \xi^3 - z\xi^2 - y\xi - x$ is expressed by the Bell polynomial as,

[1] $S_n(x,y,z) = \sum_{k=1}^{n}(k-1)!/(n-1)! * \mathrm{Y}[n,k,\{1!*z, 2!*y, 3!*x\}]$

The element sequence of any 3rd order polynomial $\xi^3 - c_2\xi^2 - c_1\xi - c_0 = \xi^3 - z\xi^2 - y\xi - x$ is expressed by the Bell polynomial as,

[2] $\mathcal{E}_n(x,y,z) = \sum_{k=0}^{n-2}(k-2)!/(n-2)! * \mathrm{Y}[n-2,k,\{1!*z, 2!*y, 3!*x\}]$

Using the Perrin sequence as the parent equations [1,2] become,

[3] $S_n(1,1,0) = \sum_{k=1}^{n}(k-1)!/(n-1)! * \mathrm{Y}[n,k,\{1!*0, 2!*1, 3!*1\}]$

[4] $\mathcal{E}_n(1,1,0) = \sum_{k=0}^{n-2}(k-2)!/(n-2)! * \mathrm{Y}[n-2,k,\{1!*0, 2!*1, 3!*1\}]$

The corresponding sequences are calculated from [3,4,], respectively

$S_{1\text{-}25}(1,1,0)$ = {0,2,3,2,5,5,7,10,12,17,22,29,39,51,68,90,119,158,209,277,367,486,644,853,1130, …}

$E_{1\text{-}25}(1,1,0)$ = {0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114,151,200,265, … }

Certain third order equations have the following property between the parent and element sequence.

[5] $\mathcal{E}_n(x,y,z)$ = Round[r* $S_n(x,y,z)$ for n> n_m

The minimum integer n_m can vary between different coefficients (c_0, c_1, c_2).

As shown in a previous chapter the value of r can be found from the real solution to the following general equation:

[6] $\mathbf{c_0}*\xi^3 - \mathbf{fc}*\xi^2 + \mathbf{D} = 0$ $r = 1/\xi$

where D is the discriminant (D < 0) of the equation $\xi^3 - c_2\xi^2 - c_1\xi - c_0$ and fc is an integer.

For Perrin's sequence $\mathbf{c_0} = 1$, fc = -1 and D = -23 with r = 0.3106288296404670777619027… For large n the number of significant figures for r can be increased to provide accuracy at large Perrin numbers. Calculating the RHS of [5] and for $n_m = 8$

$E_{8\text{-}25}(1,1,0)$ =
$\{\{3\},\{4\},\{5\},\{7\},\{9\},\{12\},\{16\},\{21\},\{28\},\{37\},\{49\},\{65\},\{86\},\{114\},\{151\},\{200\},\{265\},\{351\}\}$

In agreement with the Padovan sequence.

As mentioned in a previous Chapter the modified Bell Polynomial is equivalent to the Inter Sequence Polynomials. This can be shown in an example for N = 10. The ISP[x,y,z] was determined to be the following polynomial;

[7] ISP10[x,y,z] = $15x^2y^2 + 2y^5 + 10x^3z + 40xy^3z + 60x^2yz^2 + 25y^4z^2 + 100xy^2z^3 + 25x^2z^4 + 50y^3z^4 + 60xyz^5 + 35y^2z^6 + 10xz^7 + 10yz^8 + z^{10}$

Equation [1] for the parent sequence with n = 10 is,

[8] $S_{10}(x,y,z) = \sum_{k=1}^{10} (k-1)!/(10-1)! * \mathrm{Y}[10,k,\{1!*z, 2!*y, 3!*x\}]$

Mathematica is used to calculate this polynomial using the partial Bell Polynomial command and expanding

[9] $S_{10}(x,y,z) = 15x^2y^2 + 2y^5 + 10x^3z + 40xy^3z + 60x^2yz^2 + 25y^4z^2 + 100xy^2z^3 + 25x^2z^4 + 50y^3z^4 + 60xyz^5 + 35y^2z^6 + 10xz^7 + 10yz^8 + z^{10}$

[10] $S_{10}(x,y,z)$ = ISP10[x,y,z]

and which is also true for all n values.

The partial Bell Polynomial is also called the complete exponential Bell Polynomial. If one calculates only the partial Bell then [11]

[11] $\sum_{k=1}^{n} \mathrm{Y}[n,k,\{1!*0, 2!*1, 3!*1\}]$

results in the following integer sequence,

[12] $\{0,2,6,12,120,480,2520,21840,120960,937440,\dots\}$

Comparison of this sequence to OEIS sequences does not reveal any published sequence. However, if the sequence input is replaced with $\{1!*1, 2!*1, 3!*1\}$ then the following sequence is found,

[13] $\{1,3,13,49,261,1531,9073,63393,465769,3566611,\dots\}$

This is found as OEIS A118589 the coefficients of the exponential generating function: $\exp[x+x^2+x^3]$. By analogy it is shown that the sequence [12] are the coefficients of the exponential generating function:

$\exp[x^2+x^3]$. More accurately, each term n of the coefficients of the exponential function are multiplied by n!.

In general, the sequence input to the partial Bell Polynomial $\{1! * a, 2! * b, 3! * c\}$ results in coefficients of the generating function [14],

[14] a(n) = n! * coefficients of $\exp[ax+ bx^2+ cx^3]$

In detail, I show a calculation of $S_{23}(1,1,0)$ = 644 and compare to the exponential calculation.

The 23rd coefficient of $\exp[x^2+x^3]$ is 1496778195778145280. The index k in [1] is taken from 1 to n in each calculation so the coefficient of the exponential is a sum of these terms. It can be shown that not all terms are required in this summation since many are zero. I show that equation [1] can be converted to less sums as

[15] $$S_n(1,1,0) = \sum_{k=Round[\frac{n}{2}-n/6]}^{Round[\frac{n}{2}]} (k-1)!/(n-1)! * \mathrm{Y}[n,k,\{1! * 0, 2! * 1, 3! * 1\}]$$

I find for n = 23 that k ranges from 8 to 12. Calculating $\sum_{k=8}^{12} \mathrm{Y}[23,k,\{1! * 0, 2! * 1, 3! * 1\}]$ results in the 5 coefficients

$$\{512936840057241600, 897639470100172800, 85489473342873600, 712412277857280, 0\}$$

As predicted the sum of these coefficients agree with the 23rd exponential coefficient of 1496778195778145280. The ratio of these 5 coefficients to the exponential coefficient are rational numbers C1= $\{720/2101, 1260/2101, 120/2101, 1/2101, 0\}$.

The corresponding factorial coefficients for $(k-1)!/(23-1)!$ are

$$C2= \{\frac{1}{223016017416192000}, \frac{1}{27877002177024000}, \frac{1}{3097444686336000}, \frac{1}{309744468633600}, \frac{1}{28158588057600}\}$$

In the final calculation I find that C1 and C2 multiplied term by term and multiplied by 1496778195778145280 resulting in 644.

The partial Bell Polynomial is a combinatorial function representing the partition of elements (objects) into blocks (boxes). If the ordering is indistinguishable then this can be represented by [16].

[16] $$\mathrm{Y}[n,k,\{x1,x2,x3,x4,..\}]$$

For example 4 objects can be sorted into 2 boxes as $\mathrm{Y}[4,2,\{x1,x2,x3,x4,\ \}] = 3x2^2 + 4x1x3$ indicating there are 3 ways to put 2 objects in 2 boxes and 4 ways to put one object in 1 box and 3 objects in a box. This partition 4 object is into 7 boxes:

(a),(b),(c),(d) to [(ab),(cd)] [(ac),(bd)] [(ad),(bc)] and [(a),(bcd)] [(b),(acd)] [(c),(abd)] [(d),(abc)]

If ordering is distinguishable then this is represented by [17].

[17] $$\mathrm{Y}[n,k,\{1! * x1, 2! * x2, 3! * x3, 4! * x4,..\}]$$

Using the same example, $\mathrm{Y}[4,2,\{1! * x1, 2! * x2, 3! * x3, 4! * x4\}] = 12x2^2 + 24x1x3$. It can be shown that by different ordering in each box there are 12 ways to represent the 3 boxes and 24 ways to represent the 4 boxes.

If the x1, x2, x3 are given the value 1 then the Bell polynomial gives the number of partitions (e.g. 7 in the example above). The Bell number is defined for all possible partitions of n objects. In this example the Bell Number of 4 objects is 15.

Bell Number (4) = $\sum_{k=1}^{4} \Upsilon[4, k, \{1,1,1,1\}]$ = 15 = 1 + 7 + 6 + 1 as k goes from 1 to 4.

It was observed by Furedi[1] that the graph consisting of a cycle of n vertices has a total number of maximal independent sets given by the nth term of the Perrin sequence. This demonstrates that the partitioning of the Perrin sequence is related to the method of partitioning calculated from the modified Bell Polynomial. This partitioning is a mapping from one set of objects to another. The Perrin sequence is as expected from the above examples to represent a distinct ordering of objects. This was demonstrated in my chapter on Maximal Independent Sets, Cycles and Spanning Trees.

Recently Ricci[2] describes higher order Bell Polynomials with application to the mapping of inputs to a single output in various steps. If we define H(n,m) as the number of mappings of n inputs to a single output in m steps, it was shown that H is described by a general Bell Polynomial. The Bell Number can also be represented in *Mathematica* by a generalized Bell Polynomial. Using the example above

Bell Number (4) = $\Upsilon[\{\{1,1\},\{1,1\},\{1,1\},\{1,1\}\}]$ = H(4,2) = 15

In terms of Ricci this is also H(4,2) the number of mappings of 2 inputs to a single output in 4 steps. A graphic example is provided by Hogg and Huberman[3]. Let us calculate H(3,2) the number of mappings of 3 inputs to a single output in 2 steps.

H(3,2) = $\Upsilon[\{\{1,1\},\{1,1\},\{1,1\}\}]$ = 5

This is shown graphically from Hogg and Huberman (pg 2340)

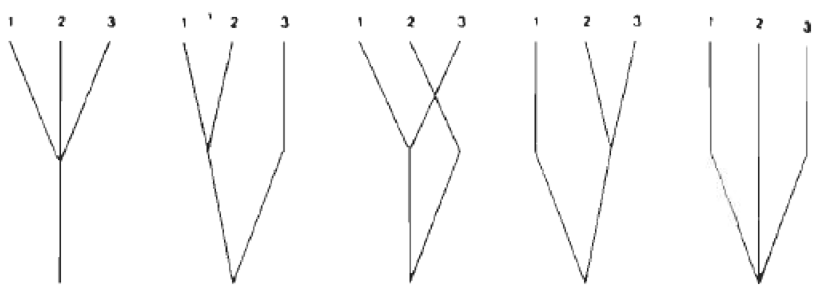

FIG. 4. The five maps of three inputs into a single output in two steps represented as trees.

Higher order mappings can be expressed by the generalized Bell number where n is the number of bracketed sets and m is the number of elements in the bracket. The number of mappings of 6 inputs to a single output in 5 steps is.

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[1] Z. Furedi, The number of maximal independent sets in connected graphs. J. Graph Theory 11 (1987), 463-470.

[2] P. Ricci and P. Natalini, Integer sequences connected with extensions of the Bell Polynomial J. Integer Sequences 20 (2017)

[3] T. Hogg and B.P. Huberman, Attractors on finite sets: The dissipative dynamics of computing structure, Physical Review A 32 (4) 1985 2338-2346.

H(6,5) = Y[{{1,1,1,1,1},{1,1,1,1,1},{1,1,1,1,1},{1,1,1,1,1},{1,1,1,1,1},{1,1,1,1,1}}] = 44590

Table 1 in reference (3) shows values for 1< n =7 and 1< m =7. Note that m = 2 is the normal Bell number. It is also shown by Ricci that H(3, m) are the pentagonal numbers OEIS A000326 and equivalent to the calculation m(3m-1)/2. By contrast, the Euler pentagonal numbers are given by m (3m (+/-) 1)/2 and their sequence shown in OEIS A001318. The second pentagonal number m (3m+1)/2 (OEIS A005449) can be indirectly obtained from the Bell polynomial.

The cycle index of a symmetric group [n] can be calculated from the Bell Polynomial. Equation [18] provides an equation for each n.

[18] $Z[n\_] := (1/n!) * \sum_{k=1}^{n} Y[n, k, \{0! * x1, 1! * x1, 2! * x1, 3! * x1, 4! * x1, 5! * x1, 6! * x1\}]$

When Z[2] = $\frac{x1}{2} + \frac{x1^2}{2}$ values of the cycle index can be calculated for integer values X1 = 1,2,3,4,…

This sequence is {1,3,6,10,15,21,28,36,45,55,66,78,91,105,..}. A second vector can be obtained from the second row of the Bell Triangle. This triangle is created using the Bell Number on the right diagonal and shifting by one level down the Bell Number on the left diagonal. The number to the right is the sum of the left number and the upper left number. The first 3 rows are (1), (1,2), (2,3,5). In a previous Chapter on Delannoy numbers I showed these numbers can be calculated from the scalar product of the values of the cycle index and the rows Pascal's triangle. If we move along the cycle index sequence and multiply be the second row of the Bell triangle we get,

{1,3}.{1,2} = 7

{3,6}.{1,2} = 15

{6,10}.{1,2} = 26

{10,15}.{1,2} = 40

These values continue to give the series for the second pentagonal numbers!

The generalized Bell Polynomial is found in the geometry of the class of Soddyian triangles. These triangles are formed from midpoints of three intersecting circles. Details are found in Jackson[4]. All primitive integer sided Soddyian triangles whose area are also integral are generated. For example, the smallest area is found in the isosceles triangle with sides (5,5,8). It is easily shown that this is a triangle of area 12. The next triangle has sides (13,40,45) and of area 252. In the reference the areas are calculated from formula [19] where m,n are integers,

[19] Soddyian triangle area STA (m,n) = $m^2n^2(m + n)^2(m^2 + mn + n^2)$

We find that (m, n) = (1, 1) and (2, 1) give the first two areas 12 and 252. When n = 1 I find that larger triangle areas are calculated from the Bell Polynomial [20]

[20] STA(m,1) = $Y[\{\{1, m, m\}, \{1, m, m\}, \{1, m, m\}\}]$

The resulting sequence for m = 1 to 10 is {12, 252, 1872, 8400, 27900, 75852, 178752, 378432, 737100, 1343100} where the 10th triangle has sides (221,12200,12221). A Bell polynomial to match [19] in the form of [20] was not found when n>1. (see note (*))

[4] F.M. Jackson, Soddyian Triangles Forum Geometricorum 13, (2013) 1-6.

The generalized Bell can also represent Perrins equation x^3-x-1 = 0.

[21] $\mathrm{Y}[\{\{-1, x, 1\}, \{x, 1,1\}\}] = -1 - x + x^3$

In this context, Bell Polynomials create the parent equation [21] and provide a solution to its integer sequence [3].

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Note* In a different generalized Bell form; STA(m,n) = $\mathrm{Y}\left[\left\{\{m^6, m\}, \left\{m^3, n^{5/2}\right\}, \left\{n^{1/2} * m^3/2\,, n^4\right\}, \{1/m^2 * n^6, n^2\}\right\}\right]$ provides values of area for all (m,n) and match values given in reference (4).