Algebra of the Perrin Sequence and its Convolutions

I continue this chapter devoted to the Perrin sequence but realizing its general application to all integer sequences. This chapter illustrates how the partial Bell Polynomial can express convolutions of an integer sequence. There is an inherent algebra that can be demonstrated between various new convolution sequences.

As shown in the last chapter the parent sequence of any 3rd order polynomial $\boldsymbol{\xi^3 - c_2 \xi^2 - c_1\xi - c_0 = \xi^3 - z\xi^2 - y\,\xi - x}$ is expressed by the Bell polynomial as,

[1] $S_n(x,y,z) = \sum_{k=1}^{n} (k-1)!/(n-1)! * \mathrm{Y}[n, k, \{1! * z, 2! * y, 3! * x\}]$

It will be convenient to generalize this and the subsequent equations. Let $S_n(l)$ be a sequence of order l. In the above l = 3. As the order is found to increase for various convolution sequences define $S(l)$ as,

[2] $S_n(l) = \sum_{k=1}^{n} (k-1)!/(n-1)! * \mathrm{Y}[n, k, \{1! * c_{i-1}, 2! * c_{i-2}, 3! * c_{i-3} \ldots l! * c_{i-l}\}]$

The element sequence is the first convolution of any l th order polynomial and is expressed by the Bell polynomial as,

[3] $\mathcal{E}_n(l) = \sum_{k=0}^{n-2} (k-2)!/(n-2)! * \mathrm{Y}[n-2, k, \{1! * c_{i-1}, 2! * c_{i-2}, 3! * c_{i-3} \ldots l! * c_{i-l}\}]$

I show that this equation is the first derivative divided by n and can be expressed in a more convenient form. Let mC $S_n(l)$ represent the mth convolution of the sequence $S_n(l)$. Express [3] in the Bell form as

[4] $\mathrm{mCS}_n(l) = \sum_{k=0}^{n-(m+1)} \mathrm{Binomial}[k+m-1, k] * k!/(n-(m+1))! * \mathrm{Y}[n-(m+1), k, \{1! * c_{i-1}, \ldots l! * c_{i-l}\}]$

where m = 1 for the element sequence.

It was shown in a previous chapter that equation [1] is equivalent to the first intersequence polynomials (ISP) and that the element or first convolution is the first derivative with respect to z with each element divided by n. For the Perrin sequence

[5] $1\mathrm{CS}_n(3) = \mathcal{E}_n(3) = (1/n)*d(S_n(3))/dz$

For higher order convolutions

[6] $\mathrm{mCS}_n(3) = (1/n)*d^m(S_n(3))/dz^m$

This equation is true for higher order sequences (l) but requires a different definition of the variable z. As an example, we found in the ISP case that the derivatives of the variable z are equal to the higher convolutions. Then using *Mathematica,* at n = 10 the derivative of [1] wrt z times (1/10) gives a constant plus a function of z; $S_{10}(3,z) = 5 + 17z + 30z^2 + 30z^3 + 30z^4 + 21z^5 + 7z^6 + 8z^7 + z^9$. It is shown that this same equation in z is obtained for [4] with m = 1 and n =11 (a shift in the first convolution sequence). The important constant {5} is the correct value of the element Padovan sequence where $z = c_2 = 0$.

In Chapter 43, I showed that the second derivative (second convolution) of the parent (Perrin) sequence can be expressed as a sequence from the following 6th order equation: $-1 - 2x - x^2 + 2x^3 + 2x^4 - x^6 = 0$

This equation is derived from the coefficients obtained by squaring the cubic equation $-(1+x-x^3)^2$ and taking its inverse. This is a general method for finding the second convolution sequence of any sequence. For a cubic equation there are 6 coefficients, and for the Perrin cubic equation the coefficient of the monic term x^5 is zero. Expressing this sequence in terms of the Bell polynomial we find

[7] $1CS_n(6) = \sum_{k=0}^{n-(2)} \text{Binomial}[k+1-1,k]*k!/(n-(2))!*\text{Y}[n-(2),k,\{1!*0,2!*2\ldots 6!*-1\}]$

I find that $1CS_n(6) = \{0,1,0,2,2,3,6,7,12,17,24,36\ldots.\}$ and 2* $S_n(3) = S_n(6)$.

From previous chapters on ISPs we notice that this is the same sequence as the second derivative of the Perrin sequence $2CS_n(3) = (1/n)*d^2(S_n(3))/dz^2$ from [6].

This first example suggests that since $1CS_n(6) = 2CS_n(3)$ we can propose the following

Proposition 1- Given the order of a sequence hierarchy of orders l_1 and l_2, the sequences obtained from the m_1 and m_2 convolutions are algebraically equivalent when $l_1*m_1 = l_2*m_2$.

I define the sequence hierarchy as the number of coefficients in the expansion of $(c_0+c_1x+c_2x^2- x^3)^a$ as 'a' increases and where a is a positive integer. The proposition is demonstrated in the following examples:

[8a,b] $2CS_n(6) = 4CS_n(3)$ and $3CS_n(6) = 6CS_n(3)$

For a given order, convolution is additive, that is m_2 is partitioned. The convolution of two sequences is defined as $A*B[n]$ where for a given order *l*,

[9] $$A*B[n] = \sum_{k=1}^{n} A[[k]]*B[[n+1-k]]$$

In the above example the 6th convolution $6CS_n(3)$ can be obtained from the following partitioning of{6} = {5+1, 4+2, 3+3} or $6CS_n(3) = 5CS_n(3)*1CS_n(3) = 4CS_n(3)*2CS_n(3) = 3CS_n(3)*3CS_n(3)$.

Proposition 2- Given a sequence of order l_1 and the m_1th convolution, the sequences are algebraically equivalent when the products $l_1m_1 = l_1m_2 + l_1m_3$ and m_1 has integer partition $m_2 + m_3$.

Continuing the convolution hierarchy, let the exponent a = 3, then $(1+x- x^3)^3 = 1+3x+3x^2-2x^3-6x^4-3x^5+3x^6+3x^7-x^9$ and the order of the sequences is nine. The sequence is now 3 times the Perrin sequence and it's first convolution is,

[10] $1CS_n(9) = \sum_{k=0}^{n-(2)} \text{Binomial}[k+1-1,k]*k!/(n-(2))!*\text{Y}[n-(2),k,\{1!*0,2!*3\ldots 9!*1\}]$

We find that $1CS_n(9) = \{0,1,0,3,3,6,12,16,30,45,\ldots.\}$ and 3* $S_n(3) = S_n(9)$.

The sequence can also be checked using the scalar product of the 9th order coefficients and the first 9 integers of the sequence to give the 10th integer.

[11] $\{1,3,3,-2,-6,-3,3,3,0\}.\{0,1,0,3,3,6,12,16,30\} = 45$

Again, from a previous discussion of the third convolution of the Perrin sequence I find that,

[12] $1CS_n(9) = 3CS_n(3)$

Based on Proposition 1 we anticipate that $2CS_n(9) = 3CS_n(6)$. This can be easily checked as true using the Bell Polynomial formulas [4], [7], and [10]. From [8b] $2CS_n(9) = 6CS_n(3)$. This shows that m values can be multiplied by integers (or any real number) based on the next Proposition,

Proposition 3- Given the order of a sequence hierarchy of orders l_1 and l_2, the sequences obtained from the m_1 and m_2 convolutions are algebraically equivalent when $l_1*m_1*n = l_2*m_2*n$.

For higher orders of new sequences obtained by convolution, Propositions 1, 2 and 3 can be used to find various new relations between these sequences without the need to calculate the convolutions.

For example, $6CS_n(6)$ = $3CS_n(12)$ = $12CS_n(3)$ = 4 $CS_n(9)$. Within each order further additive relations are possible, e.g. $12CS_n(3) = 9CS_n(3) * 3CS_n(3) = 2CS_n(9) * 2CS_n(9) = 6CS_n(3) * 6CS_n(3)$.

Since these convolutions are also calculated from ISPs (Intersequence polynomials) or partial Bell polynomials higher derivative might also be found between these polynomials. In the above example, if we assume a convolution is equivalent to a derivative then,

[13] $d^6(S_n(6,z)/dz^6 = d^3(S_n(12,z)/dz^3 = d^{12}(S_n(3,z)/dz^{12}$

I will prove [13] is **false** as writing out the cases for n = 16.

We first need to show how the variable z becomes intwined in the coefficients of $-(1+x-x^3)^a$. When a =1 we have the cubic equation. For the Perrin sequence z is the $c_2 = c(l\text{-}1)^{th}$ coefficient and z =0. We keep the variable z in the expansion allowing us to take z derivatives. The 16th ISP for parent can be calculated from the partial Bell Polynomial equation [2] with c(*l*-1) = z and c(*l*-2) = c(*l*-3) = 1. We find

[14] $S_{16}(3,z) = 90 + 448z + 1144z^2 + 2016z^3 + 2716z^4 + 2912z^5 + 2688z^6 + 2112z^7 + 1380z^8 + 880z^9 + 440z^{10} + 192z^{11} + 104z^{12} + 16z^{13} + 16z^{14} + z^{16}$

In my Chapter on Intersequence Polynomials I show that the first derivative is divided by n. After the second derivative of an ISP the derivative is divided by (m-1)!. Using these rules, the 12th derivative is found using the derivative function in *Mathematica df(x)/dx =f'(x)*

[15] $d^{12}(S_{16}(3,z)/dz^{12} = S_{16}''''''''''''(3,z)/(16*1*11!) = 78 + 156z + 1092z^2 + 1365z^4$

From equation [4], calculating $12CS_{16}(3)$ = 78 since z = 0. Since these are intersequence equations, the above equation is true for any value of $z = c_2$ keeping $c_1 = c_0 = 1$. For example, the 12th convolution of the Tribonacci series finds the 16th term to be 2691.

When a =2; $(1 + x + z * x^2 - x^3)^2 = 1 + 2x + x^2 - 2x^3 - 2x^4 + x^6 + 2x^2z + 2x^3z - 2x^5z + x^4z^2$

Collecting the coefficients for x^n we find that in equations [2] and [4],

[16] $\{c6 = 2z , c5 = (2 - z^2),\ c4 = (2 - 2z),\ c3 = (-1 - 2z),\ c2 = -2,\ c1 = -1\}$

From the partial Bell Polynomial [2] we find the Parent polynomial at n = 16.

[17] $S_{16}(z) = 180 + 896z + 2288z^2 + 4032z^3 + 5432z^4 + 5824z^5 + 5376z^6 + 4224z^7 + 2760z^8 + 1760z^9 + 880z^{10} + 384z^{11} + 208z^{12} + 32z^{13} + 32z^{14} + 2z^{16}$

As anticipated all coefficients are doubled from $S_{16}(3,z)$ but the polynomial is still 16th order! This turns out to require a doubling of the number of derivatizations for $d^6(S_n(6,z)/dz^6$ to agree with the mth convolution from equation [4]. This shows that equation [13] as written is not true since the 6C $S_n(6,z)$ will still require 12 derivatives, 2 for each convolution m.

From *Mathematica* it is shown that equation [15] is true if we write,

[18] $d^6(S_{16}(6,z)/dz^6 \sim S_{16}''''''''''''(6,z)/(16*2*11!) = 78 + 156z + 1092z^2 + 1365z^4$

When a = 4 we collect the 12 coefficients for $(1 + x + z * x^2 - x^3)^4$, substitute into [2] and find that all coefficients are four times $S_{16}(3,z)$ but the polynomial is 16th order. I find that

[19] $d^3(S_{16}(12,z)/dz^3 \sim S_{16}''''''''''''(12,z)/(16*4*11!) = 78 + 156z + 1092z^2 + 1365z^4$

This indicates each convolution requires 4 derivatizations of $S_{16}(12,z)$. This leads to Proposition 4;

Proposition 4- Given the order of a sequence hierarchy of orders l_1 and l_2, the sequences obtained from the m_1 and m_2 convolutions are algebraically equivalent when $l_1*m_1* = l_2*m_2*$. Although this can be written as $m_1CSn(l_1) = m_2CSn(l_2)$, the number of derivatizations with respect to variable $z = c_{(l-1)}$ is fixed by the lowest order l_n.

The lowest order is defined when $\mathbf{c_{(l-1)} = z}$ is the only coefficient in equation [2] that contains the variable z. This definition applies to any order equation of the form $x^l - c_{(l-1)} x^{(l-1)} - c_{(l-2)} x^{(l-2)} - \ldots\ldots c_{(0)}$. It also demonstrates that the number of derivatizations for each convolution is given by the power a in the expansion of $(-x^l + c_{(l-1)} x^{(l-1)} + c_{(l-2)} x^{(l-2)} + \ldots\ldots + c_{(0)}.)^a$. Then in terms of derivatizations the example from equation [13] becomes,

[20] $d^{6\,a_2} (S_n(6,z)/dz^{6\,a_2} = d^{3\,a_3} (S_n(12,z)/dz^{3\,a_3} = d^{12*a_1}(S_n(3,z)/dz^{12*\,a_1}$

where $a_1 = 1$, $a_2 = 2$, $a_3 = 4$. But, based on equation [4], $6CS_n(6) = 3CS_n(12) = 12CS_n(3)$.

An Example

In a previous chapter I discussed higher order Perrin sequences. The convolution of the parent equation $\boldsymbol{\xi^5 - \xi^3 - 1}$ will be shown using the Bell Polynomials discussed above. As a fifth order equation {2] becomes,

[22] $S_n(5) = \sum_{k=1}^{n} (k-1)!/(n-1)! * \mathrm{Y}[n, k, \{1!*0, 2!*1, 3!*0, 4!*0\ , 5!*1\}]$

$= \{0,2,0,2,5,2,7,2,9,7,11,14,13,23,20..\}$

The second convolution from [4] is,

[23] $2CS_n(5) = \sum_{k=0}^{n-(3)} \mathrm{Binomial}[k+2-1, k] * k!/(n-(3))! * \mathrm{Y}[n-(3), k, \{1!*0, 2!*1, 3!*0, 4!*0\ , 5!*1\}]$

$= \{0,0,1,0,2,0,3,2,4,6,5,12,9,20,19 \ldots\}$

We show that the first convolution of a 10th order equation is equal to [23] and find the parent sequence of the 10th order equation.

First square the 5th order equation to obtain coefficients of a 10th order sequence.

[24] $\boldsymbol{-(\xi^5 - \xi^3 - 1)^2 = -(1 + 2\,\xi^3 - 2\,\xi^5 + \xi^6 - 2\,\xi^8 + \xi^{10})}$

Using the coefficients on the RHS the first convolution is shown equal to [23],

[25] $1CS_n(10) = \sum_{k=0}^{n-(2)} \mathrm{Binomial}[k+1-1, k] * k!/(n-2)! * \mathrm{Y}[n-2, k, \{1!*0, 2!*2, 3!*0, 4!*-1\ , 5!*2, 6!*0,\ 7!*-2, 8!*0, 9!*0, 10!*-1\}]$

$= \{0,0,1,0,2,0,3,2,4,6,5,12,9,20,19 \ldots\}$

The parent 10th order sequence can be obtained from [2]

[26] $S_n(10) = \sum_{k=1}^{n} (k-1)!/(n-1)! * \mathrm{Y}[n, k, \{\{1!*0, 2!*2, 3!*0, 4!*-1\ , 5!*2, 6!*0, 7!*-2, 8!*0, 9!*0, 10!*-1\}\}]$

$= \{0,4,0,4,10,4,14,4,18,14,22,28,26,46\}$

As with the Perrin sequence this has been shown to be twice equation [22]. The sequence can also be derived from integer powers of the eigenvalues of the 10th order equation in [24].

Sequence Algebra and Linear Mappings

Linear sequences can be viewed as N-dimensional vector spaces. A homomorphism or a linear mapping relates two or more vectors. A simple example is the sequence hierarchy of Parent sequences. In the above example, we find that the square of two polynomials results in the sum of the two sequences. The Perrin sequence is an example and we can define the linear mapping of two functions to vector spaces as,

[27] F: [G(x,m)*G(x,m)] = $S_n(2m) = S_n(m) + S_n(m)$ where G(x,m) = $(1+x-x^3)$

The mapping takes the product of two cubic equations of order m into a sum of two sequences (vectors).

It is interesting that this mapping is reminiscent of the group homomorphism of the natural logarithm of integers where ln(a*b) = ln(a) + ln(b). Equation [27] can be extended to different equations of different order.

[28] F: $[G(x,m_1)*H(x,m_2)] = S_n(m_1+m_2) = S_n(m_1) + S_n(m_2)$ where $G(x,m_1)$ and $H(x,m_2)$ are monic polynomials.

It can be shown that if G[x,3] = $(1+x-x^3)$ and H[x,2] = $(1+x-x^2)$ then

[29] F: [G(x,3)*H(x,2)] = $S_n(5) = S_n(3) + S_n(2)$

where = $S_n(3)$ is the Perrin sequence and $S_n(2)$ is the Lucas sequence.

The linear mapping takes a different form with the Element sequences. As an example, we form the linear mapping of the Padovan sequence with the Fibonacci sequence. I have shown that these two sequences are the Element sequences of the Perrin and Lucas sequences, respectively.

The element sequence is the first convolution of parent and is defined in equation [3] above. It can be shown that the linear mapping is,

[30] F: $[G(x, l_1)*H(x, l_2)] = \mathcal{E}_n(l_1+ l_2) = \mathcal{E}_n(l_1) - \mathcal{E}_n(l_2)$ where $G(x, l_1)$ and $H(x, l_2)$ are monic polynomials and

$$\mathcal{E}_n(l) = \sum_{k=0}^{n-2} \text{Binomial}[k+1-1,k]*k!/(n-2)!*\text{Y}[n-2,k,\{1!*c_{i-1},\ldots l!*c_{i-l}\}]$$

or equivalently, the first derivative of the appropriate ISP.

The sequence $\mathcal{E}_n(2+3)$ is found in OEIS A129973 and is described as the difference between the Fibonacci sequence and the Padovan sequence, equation [30]. If G and H are the same polynomial then the difference $\mathcal{E}_n(x,l) - \mathcal{E}_n(x,l) = 0$. As a linear mapping this can be compared to the logarithmic division of integers; ln(a/b) = ln(a) - ln(b) where if a = b then ln[a/a] = 0. Depending on some value of a and b the difference can be positive or negative.

A more detailed version of equation [30] needs to be given for various second and third- degree polynomials. A general equation can be shown for two polynomials given by coefficients ($\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}$) and ($\mathbf{c1_0}, \mathbf{c1_1}, \mathbf{c1_2}$). If $\mathbf{c_1 = c1_1}$ and $\mathbf{c_2 = c1_2}$ and $\mathbf{c_0 \neq c1_0}$ then I find that with l = order 2 or 3,

[31] $\mathcal{E}_n(l_1+ l_2) = (1/ (\mathbf{c_0} - \mathbf{c1_0}))*(\mathcal{E}_n(l_1) - \mathcal{E}_n(l_2))$

This equation and [30] are equivalent when $(\mathbf{c_0} - \mathbf{c1_0}) = 1$. Note that when $\mathbf{c_0} = \mathbf{c1_0}$ the limit 0/0 is zero. Different forms of [31] are necessary when the coefficients differ, but the general equation usually applies when at least 2 of the coefficients in each equation are equal.

When $\mathbf{c_2 = c1_2}$ and $\mathbf{c_0 \neq c1_0}$ and $\mathbf{c_1 \neq c1_1}$ expression [31] is modified. Let t = $(\mathbf{c_0} - \mathbf{c1_0})/(\mathbf{c_1} - \mathbf{c1_1})$ then the element sequence is calculated as,

[32] $\mathcal{E}_k(l_1+ l_2) = \sum_{i=1}^{k}(-1)^{i-1}*t^i*(\text{E}_{k-i+1}[l_1]-\text{E}_{k-i+1}[l_2])/(\text{c}_0-\text{c1}_0)]$

Sequence Entanglement

Equations [27], [30] and [31] show that certain element sequences can be represented by a difference (or sum) of two different or similar element sequences. In Chapter 40 I describe discriminants and the element sequence. Some element sequences can be found by multiplying the sequence numbers in the parent sequence by a constant found from the cubic equation after rounding to give integers,

[33] $D*x^3 + fc*x + c_0 = 0$

Here D is the discriminant of the cubic or quadratic equation $\mathbf{z^3 - c_2\, z^2 - c_1\, z - c_0}$, fc is found in equation [6] of the chapter and the real solutions of [33] are the desired proportional constant(s).

An interesting problem occurs when equation [33] has three real solutions. There are then three potential element sequences for a single parent sequence. As we previously demonstrated the Perrin equation (1,1,0) has only one real solution to [33] and the constant defines the element Padovan sequence. The situation changes for the equation (1,2,0) which has discriminant of 5 and fc = -4. The proportional constants from [33] are $\{x_1,x_2,x_3\} = \{-1, \frac{1}{10}(5+\sqrt{5}), \frac{1}{10}(5-\sqrt{5})\}$. As mentioned in Chapter 40 there are many situations in which [33] does not give a constant for the element sequence. None of the three proportional constants when multiplied by the integers of the parent sequence and rounded give the element sequence to (1,2,0}.

[34a,b,c,d,e] Parent sequence: $\{0,4,3,8,10,19,28,48,75,124,198,323,520,844,1363, \ldots\}$

Element sequence: $\{1,4,4,9,12,22,33,56,88,145,232,378,609,988,1596, \ldots\}$

Round(x_2* Parent): $\{0,3,2,6,7,14,20,35,54,90,143,234,376,611,986, ..\}$

Round(x3* Parent): $\{0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987, ..\}$

Round((x_3/x_2)*Parent): $\{0,2,1,3,4,7,11,18,29,47,76,123,199,322,521,843, ..\}$

Note that [34d] and [34e] form the Fibonacci and its parent Lucas sequences. Equation [34c] is found in OEIS as A007492 the Fibonacci(n) - $(-1)^n$ series. Equation [34b] is the element sequence identified as A008346 the Fibonacci(n) + $(-1)^n$ series. The parent series [34a] can also be expresses as A099925 the Lucas (n) + $(-1)^n$ series.

Since we have generated two element sequences [34b] and [34d] we can examine the difference in these sequences as in [31] since $c_0 = 1$ and $c1_0 = 0$ respectively for the two sequences. Using the full sequences, I find:

[35] $\{0,2,1,4,4,9,12,22,33,56,88,145,232,378,609,988,1596,2585\} -$ $\{0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597\}$=

[34b] $\{0,1,0,2,1,4,4,9,12,22,33,56,88,145,232,378,609,988\}$

If the element is shifted by 2 units [34c] is generated by adding these sequences,

[36] $\{0,2,1,4,4,9,12,22,33,56,88,145,232,378,609,988,1596,2585\} +$ $\{0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597\}$ =

[34c]] $\{0,3,2,6,7,14,20,35,54,90,143,234,376,611,986,1598,2583,4182\}$

Equation [30] does not work in this case where G[x,3] = $(1+2x-x^3)$ and H[x,2] = $(1+x-x^2)$ and the product is $1+3x+x^2-3x^3-x^4+x^5$. Finding the element of the 5th order product,

[37] $\mathcal{E}_n(l) = \sum_{k=0}^{n-2} \text{Binomial}[k+1-1,k]*k!/(n-2)!*\text{Y}[n-2,k,\{1!*1,2!*3,3!*-1,4!*-3,5!*-1\}]$

yields a new sequence,

[38] $\{0,1,1,4,6,14,24,47,83,152,268,476,832,1453,2517, ..\}$

Which does not match any of the element sequences above. However, as a linear recurrence the 5th order equation does produce [34b,c and d] with the correct initiators in *Mathematica*,

[34b] $\text{LinearRecurrence}[\{1,3,-1,-3,-1\},\{0,1,0,2,1\},20]$ = $\text{LinearRecurrence}[\{0,2,1\},\{0,1,0\},20]$

[34c] $\text{LinearRecurrence}[\{1,3,-1,-3,-1\},\{0,3,2,6,7\},20]$ = $\text{LinearRecurrence}[\{0,2,1\},\{0,3,2\},20]$

[34d] $\text{LinearRecurrence}[\{1,3,-1,-3,-1\},\{0,1,1,2,3\},20]$ = $\text{LinearRecurrence}[\{0,2,1\},\{0,1,1\},20]$

I label this behavior of a sequence as element sequence entanglement since it mimics the behavior of classical entanglement. Given the object [34b] two outcomes are possible [34e] or [34c] depending on the sign of the object [34d]. If the Fibonacci is added to [34b] then the outcome is [34c]. If the Fibonacci is subtracted from [34b] then the outcome is a shifted right [34b] == [34e]. For this type of entanglement to occur the discriminant equation [33] must have 3 distinct real solutions for a cubic equation $\mathbf{z^3 - c_2 z^2 - c_1 z - c_0}$ which produce 2 or 3 element sequences. All element linear recurrences are also reproduced by a higher order equation of 5th or 6th order using different initiators.

R. Turk May 25, 2020

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