Elliptic Functions and the Bell Polynomial

In the previous Chapters the Bell Polynomials were shown to represent and calculate integer sequences and the polynomials represented by these sequences. In this Chapter we examine another application of the Bell polynomial in the representation of Jacobi elliptic functions. The inverse of the incomplete elliptic integral of the first kind is known as the Jacobi elliptic sine function. Corresponding to the trigonometric sin and cosine functions there are a total of 12 Jacobi elliptic functions. The elliptic sine is the “father” of all these functions since it is shown that they can all be derived from the elliptic sine.

The partial Bell polynomial has been shown by Masson to provide a formula for the inverse of the elliptic integral of the first kind[1]. A new function is also derived for the inverse for the integral of the second kind, a function which does not have a previous closed form expression. Although the elliptic sine and associated functions are programed in *Mathematica*, the corresponding elliptic ‘second’ sine and associated functions are not available. The geometry of the ellipse and elliptical orbits are defined by the inverse of elliptic integrals of the first and second kind. For example, relativistic elliptic orbits and the arc length of an elliptical orbit and the angle of this arc are calculated from the inverse of the elliptic integrals of the first and second kind[2].

In its elementary form the elliptic integrals of the first and second kind are given as:

[1] $$u = \int_0^x ((1-\mathrm{t}^2)(1-k^2\mathrm{t}^2))^{-1/2} dt$$

[2] $$w = \int_0^x](1-k^2\mathrm{t}^2)/(1-\mathrm{t}^2)]^{1/2}\, dz$$

The integration limit is 1 ≥ x ≥ 0 and the modulus k is 1 ≥ k ≥ 0.

Then the inverse of the elliptic integral of the first kind is x:

[3] sn (u, k^2) = x

Note that in the Legendre form x = Sin ϕ where ϕ is the Jacobi amplitude. If t in equation [1] is substituted with Sin θ then the elliptic integral of the first kind is

[4] $$F\ (\phi/k^2) = \int_0^\phi (1-k^2\sin^2(\theta))^{-1/2} d\theta.$$

Equation [4] is the form used for *Mathematica* for the elliptic integral of the first kind. Then the equation u = EllipticF[ϕ, k^2] calculates the same value as the integral in [1]. Equation [3] is the inverse of the elliptic integral where x = Sin ϕ from which the amplitude can be calculated using

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[1] Paul Masson,” Inverse Elliptic Integrals of the First and Second Kinds”, website https://analyticphysics.com/ (Special Functions)

[2] D.F. Lawden, **Elliptic Functions and Applications**, Chapter 5, “Relativistic Planetary Orbits” in Applied Mathematical Sciences Vol. 80, (1989), Springer Verlag, NY.

the arcsin function. From this change in variable for x the inverse of sn defined as sn $^{-1}$(x, k^2) can also be equated to the integral u = $F(\phi/k^2)$ = sn $^{-1}$(x, k^2).

In reference (1) the two elliptic integrals [1] and [2] are equated to hypergeometric functions as formulated by Appell functions of two arguments equal to t^2 and k^2*t^2. The details of inverting this hypergeometric function and expressing u as a partial Bell polynomial are left to a reader of reference (1).

Since the first and second kind integrands only differ by a power of +/- ½, the inverses of these incomplete elliptic integrals have similar expansions. The inverse functions for the first and second kind integrals are defined for sn (u, k^2) and sn2 (w, k^2), respectively. Equations [5] and [6] define sn (u, k^2) and [7] and [8] define sn2 (w, k^2).

[5] $$a1(n) = \frac{1}{2n+1} \sum_{i=\frac{n}{2}}^{\frac{n}{2}} \binom{n}{i+\frac{n}{2}} \Gamma[\tfrac{n}{2}+i+\tfrac{1}{2}] * \Gamma[\tfrac{n}{2}-i+\tfrac{1}{2}] / (\Gamma[\tfrac{1}{2}] * \Gamma[\tfrac{1}{2}]) * k^{2*(n/2-i)}$$

[6] $$sn(u, k^2) = u + \sum_{n=1}^{\infty} \frac{u^{2n+1}}{(2n+1)*n!} * \sum_{j=1}^{n} ((-1)^j * \frac{\Gamma[2n+j+1]}{\Gamma[2n+1]}) * B[n, j, \{a1[1], a1[2], a1[3], a1[4], \ldots, a1[n]\}]$$

[7] $$a2(n) = \frac{1}{2n+1} \sum_{i=\frac{n}{2}}^{\frac{n}{2}} \binom{n}{i+\frac{n}{2}} \Gamma[\tfrac{n}{2}+i+\tfrac{1}{2}] * \Gamma[\tfrac{n}{2}-i-\tfrac{1}{2}] / (\Gamma[\tfrac{1}{2}] * \Gamma[-\tfrac{1}{2}]) * k^{2*(n/2-i)}$$

[8] $$sn2(w, k^2) = w + \sum_{n=1}^{\infty} \frac{w^{2n+1}}{(2n+1)*n!} * \sum_{j=1}^{n} ((-1)^j * \frac{\Gamma[2n+j+1]}{\Gamma[2n+1]}) * B[n, j, \{a2[1], a2[2], a2[3], a2[4], \ldots, a2[n]\}]$$

where $\binom{n}{i+\frac{n}{2}}$ is the binomial and Γ is the Gamma function. B[] represents the partial Bell polynomial of n terms.

Notice in [6] and [8] that the functions sn and sn2 are a summation away from u or w. This difference can be either positive or negative. Also, the summation is only taken for n and j based on the number of arguments a1[n] or a2[n] calculated in B[]. Agreement of [5] and [6] to the *Mathematica* sn(u, k^2) expressed as JacobiSN[u,k^2] to x = Sin ϕ requires about 28 terms (n = 28) for agreement to > 70 decimal places. When n = 4 an expansion of order u^9 is obtained by expanding and simplifying equations [6] and [8] in *Mathematica*. (see reference [1] equations 20 and 21.)

Simple integration and derivatization of sn and sn2 is an advantage when expressed in the Bell polynomial form. Both u and powers of u are easily integrated or derivatized. For the first derivative with respect to u, I find the following.

[9] $$sn'(u, k^2) = 1 + \sum_{n=1}^{\infty} \frac{u^{2n}}{n!} * \sum_{j=1}^{n} ((-1)^j * \frac{\Gamma[2n+j+1]}{\Gamma[2n+1]}) * B[n, j, \{a1[1], a1[2], a1[3], a1[4], \ldots, a1[n]\}]$$

In Lawden (section 2.5) he provides several formulae for the derivatives of Jacobi elliptic functions. For the sn function,

[10] sn '(u, k^2) = cn(u, k^2) *dn(u, k^2)

The elliptic function dn is one of 12 elliptic functions. Integration of dn(z, k^2) in the limits of 0 to u provides the value of the Jacobi amplitude ϕ. Equation [10] also provides a basis for obtaining dn from the partial Bell polynomial. Since $sn^2 + cn^2 = 1$, let BellD(u) equal the value of the summation on the rhs of equation [9] and let Bell(u) equal the value of the summation on the rhs of equation [6]. Then it is possible to represent all elliptic functions with the Bell polynomial. Starting with sn the reciprocal function is 1/sn = ns and this nomenclature is used for the other functions.

[11] sn = $\mathrm{Bell(u)} + \mathrm{u}$

[12] ns = 1/ $(\mathrm{Bell(u)} + \mathrm{u})$

[13] cn = $(1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2}$

[14] nc = $1/(1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2}$

[15] dn = $(1 + \mathrm{BellD(u)}) * (1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{-1/2}$

[16] nd = $(1 + \mathrm{BellD(u)})^{-1} * (1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2}$

[17] sc = $(\mathrm{u} + \mathrm{Bell(u)}) / (1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2}$

[18] cs = $(1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2} / (\mathrm{Bell(u)} + \mathrm{u})$

[19] cd = $(1 - (\mathrm{Bell(u)} + \mathrm{u})^2) / (1 + \mathrm{BellD(u)})$

[20] dc = $(1 + \mathrm{BellD(u)}) / (1 - (\mathrm{Bell(u)} + \mathrm{u})^2)$

[21] ds = $(1 + \mathrm{BellD(u)}) / (\mathrm{u} + \mathrm{Bell(u)})\, (1 - (\mathrm{Bell(u)} + \mathrm{u})^2)^{1/2}$

[22] sd = $(\mathrm{u} + \mathrm{Bell(u)})\, (1 - (\mathrm{Bel(u)\,l} + \mathrm{u})^2)^{1/2} / (1 + \mathrm{BellD(u)})$

The integral of sn is also expressed by the Bell polynomial. For integration of sn from 0 to u1,

[23] $\frac{u1^2}{2} + \sum_{n=1}^{\infty} \frac{u1^{2n+2}}{(2n+2)(2n+1)*n!} * \sum_{j=1}^{n} ((-1)^j * \frac{\Gamma[2n+j+1]}{\Gamma[2n+1]}) * B[n, j, \{\mathrm{a1}[1], \mathrm{a1}[2], \mathrm{a1}[3], \mathrm{a1}[4], \ldots, \mathrm{a1}[n]\}]$

The equation agrees with integration of the *Mathematica* expression JacobiSN[z, k^2].

The derivative and integral for the inverses of the elliptic integral of the second kind can be easily derived from equation [8] and using equation [7].

[24] sn2' (w, k^2) = $1 + \sum_{n=1}^{\infty} \frac{w^{2n}}{n!} * \sum_{j=1}^{n} ((-1)^j * \frac{\Gamma[2n+j+1]}{\Gamma[2n+1]}) * B[n, j, \{\mathrm{a2}[1], \mathrm{a2}[2], \mathrm{a2}[3], \mathrm{a2}[4], \ldots, \mathrm{a2}[n]\}]$

The integral of sn2(w, k^2) from 0 to w1 is then given by,

[25] $\frac{w1^2}{2}+\sum_{n=1}^{\infty}\frac{w1^{2n+2}}{(2n+2)(2n+1)*n!}*\sum_{j=1}^{n}((-1)^j*\frac{\Gamma[2n+j+1]}{\Gamma[2n+1]})*B[n,j,\{a2[1],a2[2],a2[3],a2[4],...,a2[n]\}]$

In Chapter 31, *Elliptic Function and the Ramanujan Octave* I show two functions that relate the modulus k to the q-octic continued fraction.

[*10a, b*] $|uo(\tau)|=\sqrt{2}\,(k*k')^{1/4}$ and $|ue(\tau)|=\sqrt{2}\,\left(\frac{k}{k'^2}\right)^{1/4}$

where k' $=\sqrt{1-k^2}$ and $|u(\tau)|$ is calculated in [10a] for odd discriminants and [10b] for even discriminants, m. If we invert [10a, 10b] for k, I find,

[26a,b] $\text{k(odd)}^2=\frac{1}{2}-\frac{\sqrt{4-uo(\tau)^8}}{4}$ $\text{k(even)}^2=1+\frac{8}{ue(\tau)^8}-\frac{4\sqrt{4+ue(\tau)^8}}{ue(\tau)^8}$

The values for k can also be found from the ratio of the complete elliptic integrals for k and k'.

[26c] $(\text{EllipticF}[\pi/2,1-\text{k}^2]/\text{EllipticF}[\pi/2,\text{k}^2])^2=m$

We find that these equations are useful in finding Weber's class invariants[3]. The Ramanujan ladder is used to find values of all the class invariants in reference (3) from the following relationships.

[26d] $\frac{G_2^4}{G_1^4}=\frac{edge_1^2}{edge_2^2}$

where G_1 and G_2 are class invariants of two different negative integer discriminants m_1 and m_2 and $edge_i$ are the associated edges calculated from $|uo(\tau)|$ or $|ue(\tau)|$

[26e] $edge^2_i=(2/9)^{-1/3}*(|u(\tau)|)^{4/3}$

[26f] $G_i=2^{1/12}/|u(\tau)|^{1/3}$

Given the value of the q octic $|u(\tau)|$ equations [26a,b] can be substituted for k^2 in [5] or [7] above. This substitution shows through the Bell polynomial a relation between the 12 elliptic functions and the q-octic continued fraction for a given integer m. (see Chapters 29-31 for details)

A Couple of Examples

There are many examples of the use of the elliptic sine function and its counterpart sn2 found in reference (2) above. Chapters 4 and 5 of Lawden's book discuss the geometrical and physical application of the elliptic integrals. An interesting application is relativistic orbits of planets and comets. An example for comets is also found in a journal article by Sir Charles Darwin[4]. The Einstein equations for the equations of motion under Newtonian and non-Newtonian fields are derived in both references (2) and (4). Details of deriving the equation for orbital radius and angle from the elliptic integral of the first kind are found in Lawden.

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[3] H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.

[4] C. Darwin, *The Gravity Field of a Particle*, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, Vol. 249 No. 1257 (Jan 1, 1959), pp. 180-194.

An example for the motion of a comet around the sun is shown in equation [*16*] of reference (4).

[*16*] $$u = \frac{-1}{12m} + \frac{1}{3m} * (\mathrm{sn}(\xi, k^2))^2$$

where m is an arbitrary dimensionless field constant (orbital perihelion is 4m) and $\xi = \frac{1}{2} * \theta * \sqrt{\frac{5}{24}}$ is the orbital angle in radians and u = 1/r the orbital radius. In the graphs below the elliptic sn function is calculated from equation [6] with n = 20 terms for a1[n] in the partial bell polynomial. These graphs agree with similar calculations using the *Mathematica* programmed function JacobiSN[u, k^2]. The accuracy of [6] is dependent on the number of terms n and the precision of the inputs for k and u. The polynomial with 20 terms result in powers of u^{57} and k^{56} with required inputs of 200 significant figures. At higher values of θ (above 2π) the calculation of u may become chotic and lead to erroneous orbits. This can be avoided in the examples below using a judicious choice of the interval of the orbital angles. For example, the orbital angles in the perihelion of the planet Mercury represent orbits over time and the precession of Mercury as predicted by Einstein cannot be graphically observed until the value of θ is over 16 π radians.

Figure 1 demonstrates the comet trajectory of equation [*16*] over the orbital angle of stability. Above 2*1.19 radians = 217 degrees the comet shows a *semi-hyperbolic* orbit, but the comet does not complete a full orbit and shoots off to infinity at an angle of 70 degrees.

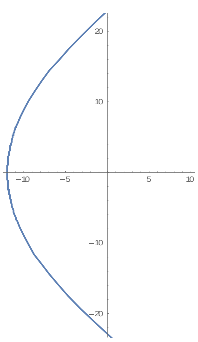

Figure 1 showing the partial trajectory of a comet under non-Newtownian gravitation before escape. Polar plot of (r, θ) with values given by Darwin in reference (3). θ ranges from -1.19*2 to 1.19*2 radians about perihelion. r is calculated from [6] as

$$\boldsymbol{r} = \mathbf{1/((-1/(12*}\boldsymbol{m}\mathbf{)) + (1/(3*}\boldsymbol{m}\mathbf{)) * ((1/2) * \sqrt{5/24} *}\boldsymbol{\theta}\mathbf{ + Bell[(1/2) * \sqrt{5/24} *}\boldsymbol{\theta}\mathbf{])^2)}$$

with $k^2 = 4/5$.

Planetary orbits can also be described by equation [*16*]. Lawden derives the formula based on two astronomical constants for the advance of the perihelion of Mercury due to relativistic effects only. The dimensionless constants are e the eccentricity of the elliptical orbit (0< e <1) and α the gravitational force term arising in Einstein's equation that is scaled on the product of the planet's angular momentum per mass and the speed of light. This term is largest for Mercury and is assigned the value of $\alpha = 5.09 * 10^{-8}$. The resulting equation is like Duffing's non-linear oscillator equation which also can be solved with elliptic functions. (discussed in reference (2))

The value of α being small, allows expansion of the oscillator to order (α^2). The resulting equation for u = 1/r is similar in form to that in reference (4),

[27] $$u = (1 - e - \frac{(1-e)^3\alpha}{2e}) + (2e + \frac{\alpha}{e} + 3e\alpha) * (\mathrm{sn}(\frac{1}{2} * \theta * \sqrt{(1 - 3\alpha + e\alpha)}, 2e\alpha^2))^2$$

Using the Bell polynomial formula for the elliptic sine function with e = 0.205 we find a nearly circular orbit that advances very slowly. In fact, the perihelion (precession) only advances by 43 seconds of arc per century. (the above formula calculates approximately 41 seconds based on the period of Mercury's orbit of 88 days.). Figure 2 shows an exaggerated advance based on e = 0.6 and a value of $\alpha = 5.09 * 10^{-3}$ calculated after 2 orbits.

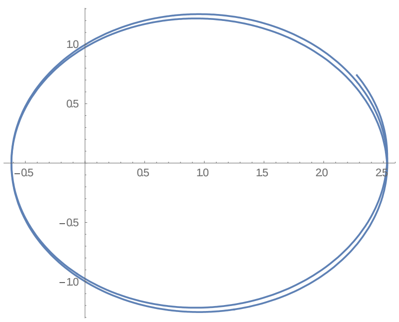

Figure 2 showing the exaggerated orbit and advance of perihelion of Mercury under non-Newtownian gravitation. Polar plot of (r, θ) with values given above and calculated using the partial Bell ploynomial formula for the elliptic sin. The calculation agrees with the programmed formula for the JacobiSN function in *Mathematica*. θ is varied from -2π to 2.1π thus showing advance (upper right) to a new elliptic orbit.

For this orbit the path will continue its precession of the point of perihelion about the semi major axis until it completes a 360-degree cycle. (Due to other planetary interactions the actual precession is larger, about 575 arcseconds per century[5]. The expansion of the Bell polynomial to n =20 is not sufficient to calculate the orbit beyond 3 orbits due to the higher value of k = 2eα. This orbit is stable and maintains the elliptical orbit of Mercury. Some orbits are erratic and seem to escape but are then drawn back and captured. The comet in Figure 3 shows this behavior as the angle varies from 0 to 5π. Start orbit at (12, 0) end at (4, 5π).

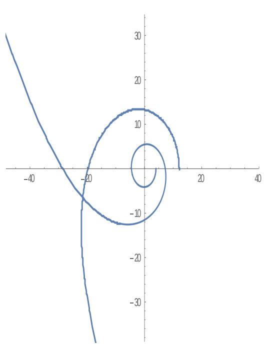

Figure 3 showing the trajectory of a comet under non-Newtownian gravitation. Polar plot of (r, θ) with values given by Darwin in reference (3). θ ranges from 0 to 5π radians about perihelion. r is calculated from [6] as

$$\boldsymbol{r} = \mathbf{1}/((\mathbf{1}/(\mathbf{12} * \boldsymbol{m})) - (\mathbf{1}/(\mathbf{3} * \boldsymbol{m})) * ((\mathbf{1}/\mathbf{2}) * \sqrt{\mathbf{1}/\mathbf{24}} * \boldsymbol{\theta} + \mathbf{Bell}[(\mathbf{1}/\mathbf{2}) * \sqrt{\mathbf{1}/\mathbf{24}} * \boldsymbol{\theta}])^{\mathbf{2}})$$

with $k^2 = 4/50$.

Note that with a smaller k value, calculations beyond 4π radians are possible.

If we integrate using equation [2] above we find that sn2[w,k^2] is the upper limit of integration x. In *Mathematica*, the elliptic integral of the second kind is programmed as EllipticE[z,k^2]. This

[5] R.S.Park, et. al, Precession of Mercury's Perihelion from Ranging to the MESSENGER Spacecraft, The Astronomical Journal, 153:121 March 2017.

upper limit is not equal to w unless we let w = EllipticE[$ArcSin[x]$, k^2]. It then indicates that use of equations [7] and [8] result in a direct calculation of x given a result from an integration involving the integral in [2].

One example of this is finding the angle of an arc section of an ellipse. Given an ellipse of eccentricity e and the length of its semi major axis the formula for the length of the arc subtended at an orbital angle from perihelion is given by the following equation.

[28] $$\text{Arclength} = \text{Semimajoraxis} * \text{EllipticE}[\phi, e\text{^}2]$$

where ϕ = $ArcSin[x]$. If we reverse the problem and ask what angle is subtended when a certain distance (arclength) is traveled then the function sn2[w,e^2] can be used to find this angle. Consider the orbit of Mercury. Since e = 0.205630 and the semi major axis (SMA) in AU is 0.387098 both based on the NASA Mercury fact sheet (on-line search), assume that the orbital distance of 0.135 AU (approximately 12.5 million miles) is traversed. Find the orbital angle. Finding the ratio of the arclength/SMA = 0.348773 = w. Equation (7) and (8) result in the value of sn2 = w + (-0.006753) = 0.342020 and $ArcSin[0.342020\ ]$ = 0.349066 radians which is equal to $\pi/9$ radians or 20 degrees.

Many other applications and problems requiring sn and sn2 are given in Chapter 5 of Lawden's book.

R. Turk

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Appendix: The n=8 expansion for Bell(u11, k^2) with u = u11 and k^2 = kea^2.

$$\begin{aligned}
\text{Bell[u11_]:=} & -\frac{\text{u11}^3}{6}-\frac{\text{kea}^2\text{u11}^3}{6}+\frac{\text{u11}^5}{120}+\frac{7\text{kea}^2\text{u11}^5}{60}+\frac{\text{kea}^4\text{u11}^5}{120}-\frac{\text{u11}^7}{5040}-\frac{3\text{kea}^2\text{u11}^7}{112}-\frac{3\text{kea}^4\text{u11}^7}{112}-\frac{\text{kea}^6\text{u11}^7}{5040}\\
& +\frac{\text{u11}^9}{362880}+\frac{307\text{kea}^2\text{u11}^9}{90720}+\frac{913\text{kea}^4\text{u11}^9}{60480}+\frac{307\text{kea}^6\text{u11}^9}{90720}+\frac{\text{kea}^8\text{u11}^9}{362880}-\frac{\text{u11}^{11}}{39916800}\\
& -\frac{11069\text{kea}^2\text{u11}^{11}}{39916800}-\frac{82913\text{kea}^4\text{u11}^{11}}{19958400}-\frac{82913\text{kea}^6\text{u11}^{11}}{19958400}-\frac{11069\text{kea}^8\text{u11}^{11}}{39916800}-\frac{\text{kea}^{10}\text{u11}^{11}}{39916800}\\
& +\frac{\text{u11}^{13}}{6227020800}+\frac{16607\text{kea}^2\text{u11}^{13}}{1037836800}+\frac{1498117\text{kea}^4\text{u11}^{13}}{2075673600}+\frac{3295067\text{kea}^6\text{u11}^{13}}{1556755200}+\frac{1498117\text{kea}^8\text{u11}^{13}}{2075673600}\\
& +\frac{16607\text{kea}^{10}\text{u11}^{13}}{1037836800}+\frac{\text{kea}^{12}\text{u11}^{13}}{6227020800}-\frac{\text{u11}^{15}}{1307674368000}-\frac{896803\text{kea}^2\text{u11}^{15}}{1307674368000}-\frac{3524081\text{kea}^4\text{u11}^{15}}{39626496000}\\
& -\frac{834687179\text{kea}^6\text{u11}^{15}}{1307674368000}-\frac{834687179\text{kea}^8\text{u11}^{15}}{1307674368000}-\frac{3524081\text{kea}^{10}\text{u11}^{15}}{39626496000}-\frac{896803\text{kea}^{12}\text{u11}^{15}}{1307674368000}\\
& -\frac{\text{kea}^{14}\text{u11}^{15}}{1307674368000}+\frac{\text{u11}^{17}}{355687428096000}+\frac{1008907\text{kea}^2\text{u11}^{17}}{44460928512000}+\frac{147498251\text{kea}^4\text{u11}^{17}}{17784371404800}\\
& +\frac{842002219\text{kea}^6\text{u11}^{17}}{6351561216000}+\frac{54822510947\text{kea}^8\text{u11}^{17}}{177843714048000}+\frac{842002219\text{kea}^{10}\text{u11}^{17}}{6351561216000}\\
& +\frac{147498251\text{kea}^{12}\text{u11}^{17}}{17784371404800}+\frac{1008907\text{kea}^{14}\text{u11}^{17}}{44460928512000}+\frac{\text{kea}^{16}\text{u11}^{17}}{355687428096000}
\end{aligned}$$