Appendix to Chapters 28-32 on Ramanujan's Class Invariants and Ramanujan's Ladder

Below are some helpful formula to find Ramanujan class invariants. Both modular and geometric equations are shown. The modular equations involve the modulus k. This modulus is used to calculate the modular q-octic continued fraction ue and uo for even and odd integers N. The remaining calculations are used to obtain the class invariants gN and GN. In previous chapters I demonstrated the geometry of the q-octic fraction to the platonic solid, the octahedron. The edge of the octahedron, its area and volume can be calculated from the q-octic. The edge is squared and labeled as edge2gN or edge2GN below.

The modulus and q-octic continued fractions. N is a positive integer representing a negative integer discriminant of a polynomial.

$$[1\,A] \quad \left(\frac{\text{EllipticF}\left[\pi/2,\, 1-k^2\right]}{\text{EllipticF}\left[\pi/2,\, k^2\right]}\right)^2 = N$$

Let k' = $\left(1-k^2\right)^{1/2}$

$$[2\,A] \qquad ue = \sqrt{2}\,\left(\frac{k}{k'^{\,2}}\right)^{1/4}$$

$$[3\,A] \qquad uo = \sqrt{2}\,\left(k * k'\right)^{1/4}$$

The Ramanujan class invariants. For each N a large G and small g invariant is calculated.

$$[4\,A] \qquad GN = 2^{1/12}/uo^{1/3}$$

$$[5\,A] \qquad gN = 2^{1/12}/ue^{1/3}$$

My definition of edge is based on the straight edge of an octahedron. The term edgeN is used for either even or odd values of N. Let edge2N represent this square of this edge for either odd or even N. The square is also denoted by edge2gN or edge2GN.

$$[6\,A] \qquad edge2gN = \left(\frac{2}{9}\right)^{-1/3} * ue^{4/3}$$

$$[7\,A] \qquad edge2GN = \left(\frac{2}{9}\right)^{-1/3} * uo^{4/3}$$

1. When N is odd

$$edge2gN = 3\sqrt{2}\,\sqrt{\frac{9}{edge2N^4} - \frac{\sqrt{81 - edge2N^6}}{edge2N^4}}$$

2. When N is even

$$edge2GN = 3\sqrt{2}\,\sqrt{\frac{-9}{edge2N^4} + \frac{\sqrt{81 + edge2N^6}}{edge2N^4}}$$

3. When N is odd or even

$$gN = \frac{\left(GN^8 + \frac{\sqrt{GN^8\,(-1+GN^{24})}}{GN^8}\right)^{1/8}}{2^{1/8}}$$

$$GN = \frac{\left(gN^8 + \frac{\sqrt{gN^8\,(1+gN^{24})}}{gN^8}\right)^{1/8}}{2^{1/8}}$$

4. For any N, odd or even these equations show edge2N/$3^{2/3}$ is the inverse of G^4.

$$GN^4 * edge2GN = 3^{2/3}$$

$$gN^4 * edge2gN = 3^{2/3}$$

5. Knowing gN or GN from the equations the Ramanujan Ladder can be used to find its large or small g counterparts. Note that as N increases the rungs of the ladder, edge2 decrease in size and the distance between rungs, G increases.

$$\left(\frac{GN}{gN}\right)^4 = \left(\frac{edge2gN}{edge2GN}\right)$$

6. For N odd or even the following equations have been described in the literature of Ramanujan’s Notebooks.

$$g4N = 2^{1/4} * gN * GN$$

$$GN = G_{1/N}$$

$$1 / gN = g_{4/n}$$

$$(gN * GN)^8 * (GN^8 - gN^8) = \frac{1}{4}$$

7. Some derived relationships between GN and edgeN. Note edge24N means the squared edge of 4*N for either small or large G

$$3\sqrt{\frac{9-\sqrt{81-edge2N^6}}{edge2N^4}}\sqrt{GN^8 + \frac{\sqrt{GN^8\,(-1+GN^{24})}}{GN^8}} = 3^{2/3}$$

$$\left(\frac{81\,GN^{16} - 9\sqrt{81-edge2N^6}\,\sqrt{GN^8\,(-1+GN^{24})}}{edge2N^4\,GN^8}\right)\hat{}\,3 = 81$$

$$\frac{edge2N * edge2gN}{2 \times 3^{2/3}} = edge24N$$

8. Let the value of a =$3^{2/3}$ then

$$\frac{a^6\,(edge2gN^2 - edge2GN^2)}{edge2gN^4\,edge2GN^4} = \frac{1}{4}$$

$$e2gN^4\,e2GN^4 = (2\,a * edge24N)\,\hat{}\,4$$

$$\frac{a^2\ (\text{edge2gN}^2 - \text{edge2GN}^2)}{(2\ \text{edge2g4N})^4} = \frac{1}{4}$$

$$\frac{a^2\ (\text{edge2gN}^1 - \text{edge2GN}^1)\ (\text{edge2gN}^1 + \text{edge2GN}^1)}{(2\ \text{edge2g4N})^4} = \frac{1}{4}$$

9. Relationships with the area of the octahedron, Ag or AG.

$$\text{Ag} = 2\sqrt{2} * \text{edge2gN}$$

$$\text{AG} = 2\sqrt{2} * \text{edge2GN}$$

$$\frac{a^2\ (\text{Ag} - \text{AG})\ (\text{Ag} + \text{AG})}{2\ \text{A4Ng}^4} = \frac{1}{4}$$

$$2\ a^2\ (\text{Ag} - \text{AG})\ (\text{Ag} + \text{AG}) = \text{A4Ng}^4$$

$$\text{gN}^4 * \text{Ag} = 2\ a\sqrt{2}$$

where the term A4Ng^4 is the 4th power of the area of a small g octahedron of integer 4*N. Note that If 4*N is an integer then N can be an integer or a fraction and the above equation also apply.
Given the values or the radical forms of GN, gN, edge2GN or edge2gN values or radical forms for many N can be obtained. In some cases knowing the radical form of three of the 4 variables will give the radical form of the fourth term. Also Mathematica can sometimes produce root approximations and radical forms after values are divided by a = $3^{2/3}$.

Complex Values

The q octic continued fraction calculates a complex value for u based on $q = e^{2\pi i\left(\frac{1+\sqrt{-N}}{4}\right)}$. The values for ue and uo are the product of a complex number and its conjugate. The equation for obtaining k and k' only provides real numbers for k and the corresponding values of ue and uo are real. The equation uses the ratio of complete elliptic integrals as shown above [1A] that is equal to N and must be solved to find k . In some cases it would be convenient to calculate the complex value of u without using the continued fraction and the complex q (nome) values.

An interesting property of the complex value of u shows that a similar ratio of complete elliptic integrals of the first kind as in [1A] can be used when the unknown modulus is u to the eighth power. The complex q octic, u_c can be found by finding the root of the following equation,

[1 C] $$\left(\frac{\text{EllipticF}[\pi/2,\ 1 - u_c{}^8]}{\text{EllipticF}[\pi/2,\ u_c{}^8]}\right)^2 * 4 = (N - 1) - 2\sqrt{-N}$$

I find that the ratio on the left equals the complex number whose real part is N-1 and imaginary part is -2 $\sqrt{-N}$. (previous literature reference?) This is only true when u_c is raised to the eight power and is the modulus of the complete elliptic integrals!

For all large G calculations, the real values uo or ue for either odd or even N is obtained from its product with its conjugate; u_c*conjugate[u_c] = uo or ue as above. Complex values of u_c for values of N (used for small g calculations) is somewhat more difficult to obtain since the real value of ue is obtained by the ratio of two q octic values, one real and the other complex. The complex value in the denominator u_{cd} can be obtained by using N/4 for N in equation [1C]. The numerator u_r can be obtained from equation [3C] to find the real real root,

[2 C] $$\left(\frac{\text{EllipticF}\left[\pi/2,\ 1-u_{cd}^8\right]}{\text{EllipticF}\left[\pi/2,\ u_{cd}^8\right]}\right)^2 * 4 = \left(N/4-1\right) - 2\sqrt{-N/4}$$

[3 C] $$\left(\frac{\text{EllipticF}\left[\pi/2,\ 1-u_r^8\right]}{\text{EllipticF}\left[\pi/2,\ u_r^8\right]}\right)^2 = N$$

It can then be shown that the complex value of u_c is given by solutions for u_r and u_{cd},

[4C] $$u_c = 2^{1/4} * u_r^{3/2}/u_{cd}$$

and the small g calculations using uo or ue are determined from u_c*conjugate[u_c] = ue or uo in agreement with [2A] and [3A]

Complex values of u determined from [1C] or [2C] have the interesting property for the modulus u_c^8 shown in [5C].

[5 C] $$\left(u_c^8-1\right) * \text{conjugate}\left[\left(u_c^8-1\right)\right] = 1$$

Let U = uo or ue be the real value from u_c * conjugate[u_c]. Then the formula for the j invariant based on the nome q above is,

[6 C] $$j\,(q) = \left(16-U^8\right)^3 / \left(108 * U^{16}\right)$$

However, if the nome q is calculated from $q1 = e^{2\pi i\left(\sqrt{-N}\right)}$ then a different complex value of u_c is obtained from the q-octic continued fraction equation. The equation for the invariant j is then

[7 C] $$j\,(q1) = \frac{16\left(u_c^{16}+14\,u_c^8+1\right)^3}{u_c^8\left(u_c^8-1\right)^4}$$

The magnitude of the resulting complex j(q1) divided by 1728 results in the same value as obtained in [6C]. The equation [7C] is converted to the octahedral equations (**1**) in v_c if $u_c = \sqrt{v_c}$ with numerator and denominator polynomials based on a projective geometry on the octahedron. Equation [25] in my Chapter 31 gives an equation converting U(q) to u_c(q1).

1. “Octahedral Equation” website, https://mathworld.wolfram/OctahedralEquation.html

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