Cycle Index of Symmetry Groups and the Jacobi Polynomial

The Jacobi polynomial and the hypergeometric function were previously shown to calculate the nth Perrin number. I will describe in this Chapter how the Jacobi polynomial can produce numbers for a class of sequences.

Consider a set of numbers generated from the following sequences: Let N1 be an odd integer.

$$\text{N2} = ((\text{N1} - g))/2 \quad \text{and} \quad \text{imax} = Floor[\frac{N2}{g}]$$

$$A = \text{RecurrenceTable}[\{a[n] == a[n-1] - g, a[1] == \text{N2}\}, a, \{n, imax + 1\}]$$

$$B = \text{RecurrenceTable}[\{b[n] == b[n-1] + 2, b[1] == 1\}, b, \{n, \{n, imax + 1\}]$$

The number g was previously described in Chapter 39 and represents order for a polynomial $x^g - x^{g-2} - 1 = 0$. When g=3 equation [1] gives the Perrin numbers for N1;

[1] $$P(n1\ odd, g) = \text{n1} * \sum_{i=1}^{\text{imax}+1} (1/(\text{A}[[i]] + \text{B}[[i]])) * \text{Hypergeometric2F1}[-\text{A}[[i]], -\text{B}[[i]], 1, 1]$$

This hypergeometric function can be transformed as

[2] $$So = \sum_{n=1}^{\text{imax}+1} (-1)^n (1/(A[[n]] + B[[n]])) * \text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), B[[n]], 3]$$

where $P(n1\ odd, g) = n1 * Abs[So]$

It is interesting to note that the integer 3 in So is superfluous. The value of $B[[n]]$ assures that So is a constant and not a polynomial.

I find that the value of individual Jacobi polynomials in [2] are

[3] $$\text{JacobiP}[A[[n]], (-(A[[n]] + B[[n]]) - 1), B[[n]], 3] = S_{A(n)}(i)$$

where $S_{A(n)}(i)$ is the ith entry of the cycle index polynomial of the symmetry group $A[[n]]$. In *Mathematica* this can be programmed as;

[4] $$\text{CycleIndexPolynomial}[\text{SymmetricGroup}[A[[n]]], \text{Table}[j * n, \text{Max}[A]], A[[n]]]$$

where i = j*n.

It can be shown that for the Perrin numbers g= 3 and j = 2 that the sum of the (imax+1) cycle indices from [4] has a special importance. If we write the vector of values of the cycle indices as

[5] CIP = $\text{Table}[\text{CycleIndexPolynomial}[\text{SymmetricGroup}[A[[k]]], \text{Table}[j * k, \text{Max}[A]], A[[k]]], \{k, 1, \text{Length}[A]\}]$

Then the sum of these numbers results in a **Padovan number!**

Let the vector

[6] AB= $(1/(\text{A}[[k]] + \text{B}[[k]]))$,

then the associated Perrin number is N1*CIP.AB where CIP.AB is a scalar dot product.

From [2] it is also noted that all sigma orbits $Abs[So]$ for the prime numbers are integers represented by the symmetry groups of $A[[k]]$. Furthermore, the product CIP.AB is an integer for prime N1.

From the above, the variable j is only required in [4] and [5] when calculating the Padovan and Perrin numbers using the cycle index sum of symmetry groups. The hypergeometric formula does not show a j value. When the value of g is changed to 5,7, or 9 I find that the associated sequence numbers calculated from [5] with j=2 the product N1* CIP.AB results in sequence numbers of the 3rd, 5th and 7th order maximal independent sets in a cycle graph (see Table in Chapter 39). These represent sequences associated with polynomials of order g for $x^g - x^{g-2} - 1 = 0$. In my discussion of Bell Polynomials and with intersequence polynomials I showed how the Padovan sequence is an element sequence of the parent Perrin sequence. This is mainly represented by the initiating g values for the linear recurrences, by convolution of the parent sequence or by taking the first derivative of the parent intersequence polynomial.

The cycle index sum of the symmetry groups CIP result in numbers from an associated element sequence represented by polynomials of order g for odd values of g, j = 2 and $x^g - x^{g-2} - 1 = 0$. The calculated parent sequence numbers are then given by N1*CIP.AB.

Remember that the g roots of $x^g - x^{g-2} - 1 = 0$ can be summed to give the parent sequence numbers. The nth power of these individual sums results in the nth sequence number. This number is also represented as vectors CIP and AB. Note that the g-2 power of x can also be expressed as the g-j power. This leads to the following observation:

The numbers in the element sequences of $x^g - x^{g-j} - 1 = 0$, for j = 1 to j = g-1, are represented by the cycle index sums of the symmetry groups CIP. These numbers can be expressed as vectors CIP(N1, g, j). Example, Padovan numbers are sums of CIP(N1,3,2) and the sum of CIP(N1,3,1) is the Narayana's cows sequence (OEIS A000930}.

Since CIP does not calculate all the numbers of the sequence but only uses odd N1, there is a gap of j numbers between each calculation. It is also required that [g, j] = 1 or that the g and j do not have a common factor. Therefore CIP[N1,9,3] and CIP[N1,9,6] do not express proper sequence numbers.

Example: CIP[37,9,2] = $\{15,56\}$ with a total = 71. For j= 2 calculate a parent sequence number AB = $\{\frac{1}{15},\frac{1}{8}\}$ so 37*CIP[37,9,2].AB =296. This number agrees with the 37th number in the linear sequence from *Mathematica*, $\text{LinearRecurrence}[\{0,1,0,0,0,0,0,0,1\},\{0,2,0,2,0,2,0,2,9\},\text{N1}]$. The first g terms of the parent and element linear recurrences can be found from the Bell Polynomials developed in Chapter 43. For the element sequence in this example the (N1+3)rd term of the linear recurrence agrees with the total from CIP[37,9,2].

For values of j not equal to 2 the parent sequence number is found not to be calculated from N1*CIP.AB. For other values of j equation [3] for the Jacobi Polynomial must be modified to include j. Logically, the term B[[k]] calculated from the recurrence table for b must be corrected. The modified equations are,

[7] $B = \text{RecurrenceTable}[\{b[n] == b[n-1] + j, b[1] == 1\}, b, \{n, \{n, imax + 1\}]$

[8] $\text{JacobiP}[A[[n]], \left(-\left(A[[n]] + B[[n]] + (j-2)\right) - 1\right), B[[n]] + (j-2), 3] = S_{A(n)}(i)$

Equation [8] then agrees with CIP[N1,g,j] for all j<g and is shown to be equal to [3] only when j=2.

Fortunately, it is now possible to find sequence numbers for the parent sequences for all j<g. Equation [6] is modified similarly as,

[9] AB= $(1/(\mathrm{A}[[k]]+\mathrm{B}[[k]]+(j-2)))$

For j = 1, I find the parent sequence numbers as N2*CIP[N1,g,1].AB. Since the Perrin numbers are N1* CIP[N1,g,2].AB, there is a shift in location of the parent sequence number by (N1+g)/2 places. In general, the parent sequence numbers for N1 are calculated as

[10] (N2+ (j-1)*(N1+g)/2)*CIP[N1,g,j].AB

where the parent number is at the (N2+ (j-1)*(N1+g)/2)[th] place.

The hypergeometric formula [1] previously derived to calculate Perrin numbers can be modified to all g and j valued sequences discussed in this chapter,

[1A] $P(n1,g,j)=(\mathrm{n2}+(\mathrm{j}-1)*\frac{\mathrm{n1+g}}{2})*\sum_{i=1}^{\mathrm{imax}+1}(1/(\mathrm{A}[[i]]+\mathrm{B}[[i]])+(j-2))*\mathrm{Hypergeometric2F1}[-\mathrm{A}[[i]],-\mathrm{B}[[i]]-(j-2),1,1]$

Partition and combinatorics of numbers are associated with the Perrin, Padovan and associated sequences of varied values of the constants of g and j. The cycle index of a symmetric group counts the number of possible colorations of its objects. As an example, consider the case for j=1 and g=3. Let N1 = 19, then A[[k]] = {8,5,2} and we look at the coloring for k = 1, 2 and 3 colors of symmetric groups of 8,5, and 2 objects. Using *Mathematica*,

[11] $\mathrm{poly1}=\mathrm{CycleIndexPolynomial}[\mathrm{SymmetricGroup}[8],\mathrm{Array}[\mathrm{Subscript}[a,\#\#]\&,13]]$

$$\frac{a_1^8}{40320}+\frac{a_1^6a_2}{1440}+\frac{1}{192}a_1^4a_2^2+\frac{1}{96}a_1^2a_2^3+\frac{a_2^4}{384}+\frac{1}{360}a_1^5a_3+\frac{1}{36}a_1^3a_2a_3+\frac{1}{24}a_1a_2^2a_3+\frac{1}{36}a_1^2a_3^2+\frac{1}{36}a_2a_3^2+\frac{1}{96}a_1^4a_4+\frac{1}{16}a_1^2a_2a_4+\frac{1}{32}a_2^2a_4$$
$$+\frac{1}{12}a_1a_3a_4+\frac{a_4^2}{32}+\frac{1}{30}a_1^3a_5+\frac{1}{10}a_1a_2a_5+\frac{a_3a_5}{15}+\frac{1}{12}a_1^2a_6+\frac{a_2a_6}{12}+\frac{a_1a_7}{7}+\frac{a_8}{8}$$

[12] $\mathrm{poly2}=\mathrm{CycleIndexPolynomial}[\mathrm{SymmetricGroup}[5],\mathrm{Array}[\mathrm{Subscript}[a,\#\#]\&,13]]$

$$\frac{a_1^5}{120}+\frac{1}{12}a_1^3a_2+\frac{1}{8}a_1a_2^2+\frac{1}{6}a_1^2a_3+\frac{a_2a_3}{6}+\frac{a_1a_4}{4}+\frac{a_5}{5}$$

[13] $\mathrm{poly3}=\mathrm{CycleIndexPolynomial}[\mathrm{SymmetricGroup}[2],\mathrm{Array}[\mathrm{Subscript}[a,\#\#]\&,13]]$

$$\frac{a_1^2}{2}+\frac{a_2}{2}$$

The number of ways to color 8 objects red, 5 objects red and green and 2 objects red, green, and blue are calculated.

[14] $\mathrm{poly1/.Subscript}[a,\mathrm{i\_}]\to r^i\mathrm{//Expand}\quad r^8$

[15] $\mathrm{poly2/.Subscript}[a,\mathrm{i\_}]\to r^i+g^i\mathrm{//Expand}\quad g^5+g^4r+g^3r^2+g^2r^3+gr^4+r^5$

[16] $\mathrm{poly3/.Subscript}[a,\mathrm{i\_}]\to r^i+g^i+b^i\mathrm{//Expand}\quad b^2+bg+g^2+br+gr+r^2$

These results correspond to 1, 6 and 6 ways of coloring for a total of 13. This agrees with the total calculated for CIP[19,3,1] from equation 5 with g=3 and j=1 which is the N2 = (N1-g)/2 nd or 8th number found in (OEIS A000930}. The associated number for the parent sequence is 21 from a calculation of equation [10], N2*CIP[19,3,1].AB. This number agrees with the 8th number in OEIS A001609. A similar interpretation for the cycle index sum can be made for other values of g and j.

Sequence Numbers for $x^g – x^{g-j} – d$

The cycle indices and values for A[[n]] and B[[n]] were obtained above for finding the sequence numbers for the equations $\mathbf{x^g – x^{g-j} – 1}$. If we assume that these values can also apply to the general class of polynomials $\mathbf{x^g – x^{g-j} – d}$ where d is any real or complex number, then it is possible to show that sequence numbers for the element and the parent sequences are obtainable by an expansion of the symmetry cycle indices in powers of d. I use the Jacobi Polynomial to find the cycle indices, defining the jacobi polynomial index as

[17] JPI = $\mathrm{Table[Abs[JacobiP}[A[[i]],(-(A[[i]]+B[[i]]+(j-2))-1),B[[i]]+(j-2),3]],\{i,1,\mathrm{imax}+1\}]$

where the absolute value assures positivity. This equation is equivalent to CIP in [5] above.

The expansion in d for the element sequences is,

[18] $$\sum_{k=1}^{\mathrm{imax}+1} d^{k*j-1} * \mathrm{JPI}[[k]]$$

The sequence number occurring at j*(N2+g)-(g-4) for any odd integer of g and where j < g.

The expansion in d for the parent sequence is,

[19] $$\mathrm{NE} * \sum_{k=1}^{\mathrm{imax}+1} d^{k*j-1} * \mathrm{JPI}[[k]] * \mathrm{AB}[[k]]$$

where AB is obtained from equation [9] above and NE = $(\mathrm{N2}+(j-1)*(\mathrm{N1}+g)/2)$. The sequence number occurs at 2N1 +g. In all calculations N1 is used to find AB and JPI from the recurrence tables for A and B found above.

Given a general equation of order g, let $\mathbf{x^g – c1*x^{g-j} – c0}$ be a polynomial with coefficients c1 and c0. Then the j*(N2+g)th and NEth sequence numbers are, respectively,

[20] $$\sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}}(\mathrm{c1}^{-g/j} * \mathrm{c0})^{k*j-1} * \mathrm{JPI}[[k]]$$

[21] $$\mathrm{NE} * \sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}}(\mathrm{c1}^{-g/j} * \mathrm{c0})^{k*j-1} * \mathrm{JPI}[[k]] * \mathrm{AB}[[k]]$$

Where N2 = $\frac{\mathrm{N1}-g}{2}$

Cycle Index Symmetry Sequence Numbers

All equations can be simplified above using the definitions for A[[i]], and B[[i]]. Given numbers N1, g, j, c0, and c1 equations [20] and 21 are in *Mathematica*,

[22] $$\sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}}(\mathrm{c1}^{-g/j} * \mathrm{c0})^{k*j-1} * \mathrm{Abs[JacobiP[N2}-(k-1)g, g(-1+k)-jk-\mathrm{N2}, j*k-1,3]]$$

[23] $\mathrm{NE} * \sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}} (\mathrm{c1}^{-g/j} * \mathrm{c0})^{k*j-1} * \mathrm{Abs}[\mathrm{JacobiP}[\mathrm{N2}-(k-1)g, g(-1+k)-jk-\mathrm{N2}, j*k-1,3]] * (1/(\mathrm{N2}-(k-1)(g)+j*k-1))$

where N2, NE and imax are previously defined. These equations can also be written in hypergeometric form as

[24] $\sum_{i=1}^{(\mathrm{imax}+1)} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}} (\mathrm{c1}^{-g/j} * \mathrm{c0})^{i*j-1} * \mathrm{Hypergeometric2F1}[-(\mathrm{N2}-(i-1)g), -(j*i-1), 1, 1]$

[25] $NE * \sum_{i=1}^{(\mathrm{imax}+1)} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}} (\mathrm{c1}^{-g/j} * \mathrm{c0})^{i*j-1} * 1/\big((\mathrm{N2}-(i-1)g)+(j*i-1)\big) * \mathrm{Hypergeometric2F1}[-(\mathrm{N2}-(i-1)g), -(j*i-1), 1, 1]$

These formula express certain sequence numbers found as sums of cycle indices for symmetric groups. The sequence numbers belong to sequences for **x^g – c1*x^{g-j} – c0** which in many cases are linear recurrences. However, they can also be found by some specialized generating functions for these sequences. Also, these formulae represent sequences in which g >j and when g and j are reversed (g<j). They are also suitable for cases when g is even. (N1 is then an even number). An interesting non-linear generating function for the element sequences [22] or [24] can be shown in *Mathematica* code as,

[26] $\mathrm{CoefficientList}[\mathrm{Series}[x^{g-2}/((1-\mathrm{c1}*x)^j - x^g), \{x, 0, (\mathrm{N2}+(g-2))\}], x]$

where the series ends at (or about) N2 + (g-2) with the equivalent sequence number to the element sequence formula. Also, the constant c0 =1 must be used but c1 can be variable. The equivalent non-linear sequence for the Parent sequences [23] and [25] is,

[27] $\mathrm{CoefficientList}[\mathrm{Series}[\frac{g-(g-j)x*\mathrm{c1}}{(1-\mathrm{c1}*x)^j - x^g}, \{x, 0, \mathrm{N2}\}], x]$

I previously indicated that when j and g have a common factor that a linear sequence could not be found. Equations [26[and [27] show equivalent sequence numbers for the element and parent sequences when g and or j are even numbers and are common factors. When g and or j are odd then the parent sequence number is obtained from a linear recurrence and can be found from the generating function, (for any c1 and c0),

[28] $\mathrm{CoefficientList}[\mathrm{Series}[-\frac{x^{(g-j)}(j*\mathrm{c1}+\mathrm{c0}*g*x^{(g-j)})}{-1+\mathrm{c1}*x^j+\mathrm{c0}*x^g}, \{x, 0, \mathrm{NE}+g-2j\}], x]$

Many of these sequences are combinatoric and represent sequences not found from expected linear recurrences. For example, g = 2 and j = 4 represents the OEIS sequence numbers A290890 and g =4 and j =2 the sequence A024490. See A290891 when g= 3 and j = 6.

In the next chapter I discuss how a general quintic equation can be solved by finding the solutions of **x^g – x^{g-j} – d = 0** when g = 5 and j = 4 and d is usually a complex number.

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