Solving the Quintic using Methods available in the 19th Century

Mathematicians in the 17th to 19th century sought many avenues to find a formula for solving a general quintic equation of the form

[1] $$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

where we assume a, b, c , d, and e are integers or real numbers after a coefficient of x^5 has been removed by division. Although history showed that formula could be found for polynomials of the second, third and even fourth order no such formula was found for the fifth order equation [1]. It took the genius of researchers such as Neils Henrik Abel, Carl Gustav Jacob Jacobi, Charles Hermite and Evariste Galois in the 19th century to prove using symmetry and group theory that such a formula involving radicals was not possible in the general case. However, during that century an important discovery of elliptic functions helped to fill in the gap of what imaginary numbers and radicals alone could not solve.

In the last chapter on the cycle index of symmetry groups I demonstrated how symmetry was used to find the integers in sequences associated with polynomials of the form $z^n - z^{(n-g)} - 1$ where n is an odd integer with the integer g < n. Although roots of these polynomials were not calculated it will be shown that symmetry also plays a role in finding these roots.

In this chapter I will illustrate how methods available to mathematicians in the 19th century can be applied to find solutions to the general quintic equation. The methods available included formula to solve quadratic equations, cubic equations, and special forms of elliptic functions derived in the mid-19th century. Even without a computer these operations were possible to compute by hand from tables or by time standard algebraic methods. Today, we have programs available to perform these operations, and I will use them only keeping the 19th century operation in mind.

Two discussions are mentioned which demonstrate the possibility of performing this feat. In the first discussion[1] a method for decreasing the number of coefficients in the general quintic is described. The *Bring-Jerrard*[2] reduced normal form for the general quintic [1] is,

[2] $$z^5 - z - a_o = 0$$

the constant a_o can be a real or complex number.

The second discussion[3] demonstrates how [2] can be solved with Jacobi Theta functions[4]. The strategy for solving a quintic [1] is 1.) reduce the general quintic using *Bring-Jerrard* to the z-form of equation [2], 2.) solve the reduced equation using Theta functions and 3.) reverse the normal form solution by transformations back to the general quintic in the variable x. With the Theta function solution of [2] this

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[1] "How to Transform a General Higher Degree Five or Higher Equation to Normal Form", Math Stack Exchange, https://math.stackexchange.com/questions/542108

[2] Bring, E.S. Quart. J. Math, 6, 1864

[3] "How to Solve Fifth Degree Equations by Elliptic Functions", Math Stack Exchange, https://math.stackexchange.com/questions/540964/

[4] Klein, F., "Vorlesungen uber die Theorie der elliptichen Modulfunctionen", 2 vols. Leipzig, Germany Teubner 1890-92.

reverse process only requires solution of quadratic equations and one cubic equation also solved by a quadratic equation!

This chapter illustrates how all five solutions of [2] are obtained from theta functions by use of symmetry and permutation. I previously explained how permutation was used from the cycle index of various symmetry groups to find sequence numbers from a reduced quintic. The symmetry of groups played a role in the proof by Galois and Abel that the quintic could not be solved by radicals alone. The Jacobi Polynomial was also shown to represent this symmetry. This chapter also demonstrates the third strategy above in detail and explains the interpretation of the results. This was not explained in the discussions found in the footnotes.

The methods will be most easily followed by using an example, otherwise it is difficult to describe the finer details. Equation [3] will be the quintic we wish to solve.

[3] $$x^5 + 2x^4 + x^3 + 3x^2 + 4x - 2 = 0$$

Start by first applying a quadratic transformation with a new variable, y, where $y = x^2 + m x + n$. Substituting y into [1] and using the *Mathematica* commands mentioned in footnote (1) one obtains an equation in y;

[4] $$-4 - 8m + 6m^2 - 2m^3 + 4m^4 - 2m^5 - 28n + 18mn - 20m^2n + 18m^3n - 4m^4n + 7n^2 - 31mn^2 + 10m^2n^2 + 3m^3n^2 + 3n^3 - 7mn^3 - m^2n^3 - 2n^4 + 2mn^4 - n^5 + (28 - 18m + 20m^2 - 18m^3 + 4m^4 - 14n + 62mn - 20m^2n - 6m^3n - 9n^2 + 21mn^2 + 3m^2n^2 + 8n^3 - 8mn^3 + 5n^4)y + (7 - 31m + 10m^2 + 3m^3 + 9n - 21mn - 3m^2n - 12n^2 + 12mn^2 - 10n^3)y^2 + (-3 + 7m + m^2 + 8n - 8mn + 10n^2)y^3 + (-2 + 2m - 5n)y^4 + y^5$$

The two variables m and n can be easily solved for by setting the coefficients of y^4 and y^3 equal to zero. Since the coefficient for y^4 is linear substituting for m or n into the y^3 coefficient only leads to a quadratic equation which can be solved by the quadratic formula. Then,

[5 a,b] $$m = \tfrac{1}{6}(51 + 5\sqrt{93}) \qquad n = \tfrac{1}{3}(9 + \sqrt{93})$$

And substitution into [4],

[6] $$\frac{-11712117888 - 1214488512\sqrt{93}}{7776} + \frac{(190471392 + 19734624\sqrt{93})y}{7776} + \frac{(52059024 + 5402160\sqrt{93})y^2}{7776} + y^5$$

Some more drastic steps are required to remove the y^2 coefficient in [6]. It is instructive to show the complete steps in algebraic form by setting the above equation in known variables u, v ,w;

[7] $$\mathrm{w} + \mathrm{v}\ \mathrm{y} + u\,y^2 * + y^5$$

and "depress" the function by substituting z for y with,

[8] $$z - (y\text{^}4 + p * y\text{^}3 + q * y\text{^}2 + r * y + s)$$

and new coefficients p, q, r, and s. The output is large and is not reported here but can be found from the *Mathematica* code;

[9] $$\mathrm{Collect[Resultant[}w + v * y + u * y^2 + y^5, z - (y\text{^}4 + p * y\text{^}3 + q * y\text{^}2 + r * y + s), y], z]$$

The result is another quintic in z but, as in [4] above it is convenient to remove the z^4 and z^3 terms by solving two equations in p, q, s and setting coefficients in r to zero in the z^3 term. These coefficients for z^4 and z^3 respectively, are;

[10a] $-5s + 3pu + 4v$

[10b] $(10s^2 + 3qru - 12psu + 3p^2u^2 - 3qu^2 + 2q^2v + 4prv - 16sv + 5puv + 6v^2 + 5pqw + 5rw - 4uw)$

[10c] $3qu + 4pv + 5w$

Where [10c] are the r terms in [10b] to set to zero. It can be seen that from [10a] p can be expressed as s and q expressed as p(s) from [10c]. The substitution in [10b] results in a **quadratic** for s.

[11] $-5.82383135572142095065 \times 10^9 + 1533275.35931742027s + s^2 = 0$

The value of s obtained from the two solutions of [11] is chosen such that the calculated p and q make [10c] equal to zero. These values are

[12a] s = $-1537064.2908477399259422416857636758642703678814 46$

[12b] p = $-196.13033794078860424909672197706602538471946038 91$

[12c] q = $1330.8785495557748839596848120978891966485659730949$

Since the coefficients of r have been chosen to remove r from the above quadratic, we substitute s, p, and q into the coefficient for z^2 to obtain a cubic in r.

[13] $6.27091637 \times 10^{17} - 4.88507761 \times 10^{13}r + 2.67654123 \times 10^9 r^2 + 13394.491820r^3 = 0$

Methods were known in the 1800 to solve cubic equations. One formula was devised by Girolamo Cardano (1501-1576) for solving the real root. The cubic is depressed to the form x^3+ax+b. I will show another formula for calculating all the roots of a depressed cubic equation. This method will be used again in a latter section. *Cardano's formula* is easier to express as a quadratic formula. First equation [13] when divided by the r^3 coefficient can be depressed by substituting y = r^2 + m3*r +1 where m3 can be found from the y^2 coefficient of the resulting equation. I obtain a value m3 = 236326.93548118549465 and the depressed equation,

[14] $5.290254966524434 \times 10^{28} - 3.6787144097417315 \times 10^{18}y + y^3$

Let R1 be equal to the y coefficient and R0 the constant coefficient of [14]. Then Cardano's quadratic is

[15] $z^2 + \mathrm{R0} * z - (\mathrm{R1}/3)^3 = 0$

The cube root of the solutions to [15] are then obtained as r_1 and r_2. The solution to [15] is $r_1 + r_2$ and should be real. Since the sign of the solutions can be +/- the solution is checked by the following conditions,

[16] $3 * r_1 * r_2 = -R1$ and $r_1^3 + r_2^3 = -R0$

The real solution to [14] is found to be y = $-4.079878779420581 \times 10^9$. The remaining two complex solutions can be obtained from the exponential form,

[17] $r_1 * e^2 + r_2 * e$ and $r_1 * e + r_2 * e^2$ where $e = e^{\frac{2i\pi}{3}}$

Any solution of [14] can be used. Choosing the first solution of [17] ($y_3 = 2.03993938971029081 \times 10^9 + 2.9673799438786146 \times 10^9 i$) after correction for sign, the solution of [13] is found from "un-depressing" equation [14] by solving for

[18] $r^2 + m3*r + 1 - y_3 = 0$

Again, one chooses the solution of [18] that satisfies [13] as

[18b] r = $8875.6738976278035 + 11678.9987149356612i$.

Substituting all values calculated for the resultant in [9] I obtain the semi depressed form of [3] (note that many significant figures are truncated),

[19] $(8.1646850 \times 10^{30} + 2.1401262 \times 10^{30}i) - (8.5837588 \times 10^{23} - 4.09226343 \times 10^{24}i)z + z^5$

Only one more step is needed to reduce this to the normal form of [2]. Define a new variable z = t/f where f is the quartic of the z coefficient,

[20] $f = (-1/(-8.58375880 \times 10^{23} + 4.092263 \times 10^{24}i))^{1/4} = 6.590449345 \times 10^{-7} + 2.33877334 \times 10^{-7}i$

After substitution, the normal equation that we seek in t is,

[21] $(-0.537481457474102994 + 1.30532369035692i) - t + t^5 = 0$.

The constant coefficient in [21] is labeled the constant d5.

Solving with Elliptic Functions

The Jacobi theta function is a function defined in two variables, a complex number z and τ a half period ratio defined in the upper half of the complex plane. The nome is defined as the quantity q = exp(π*i* τ) and plays a major role in the Jacobi theta function. I previously discussed many of the Jacobi functions including the 12 elliptic functions such as sn and cn which were used to describe astronomical orbits. Much of this research was done in the early 1800's so the techniques for using these functions to solve the quintic were certainly available.

When discussing the elliptic integral of the first kind and elliptic functions sn and cn there was a parameter k called the elliptic modulus. This modulus is closely tied to the half period ratio and the nome of the theta function. It was previously used to as a parameter related to the discriminants of cubic polynomials and in certain cases the used to find solutions to Weber's class invariants through Ramanujan's q octic continued fraction. Since Ramanujan's research occurred into the 20th century the q octic will not be used but it is shown to be equivalent to the theta function solutions of the quintic. It is remarkable that a single equation for k can be used which is only dependent on the constant d5, I derived in the last section. Define k as,

[22] $k = \text{Tan}[(1/4) * \text{ArcSin}[\frac{16}{25*\sqrt{5}*\text{d5}^2}]$

The "discriminant" p1 is then defined by the ratio of complete integrals of the first kind,

[23] $\text{p1} = i * \frac{\text{EllipticK}[1-k^2]}{\text{EllipticK}[k^2]}$

From [22] and [23] I find $k = -0.02539844850958 + 0.025382080i$ and $\text{p1} = -½ + 3.000459110800623i$.

Let j= 0, 1, 2, 3, 4 then the nome is found to be an exponential function of j as

[24] $\text{q[j\_]} := \text{Exp}[\pi * i * (1/5) * (\text{p1} + 2j)]$

I also define two other parameters, the constant q8 = $\mathrm{Exp}[2\pi * i/8]$ and an index jk = $\{0,1,2,0,1\}$ and jl = $\{0,0,0,1,1\}$. These indices are required to correctly cycle the nome as j increases. The center of the calculation is the theta function which is actually a ratio of two theta functions and programmed in *Mathematica* as,

[25] $S[\mathrm{j\_}] := (\mathrm{EllipticTheta}[2,0,q[j]]/\mathrm{EllipticTheta}[3,0,q[j]])^{1/2} * \mathrm{u8}^{\mathrm{jk}[[j+1]]} * (-1)^{\mathrm{ji}[[j+1]]}$

The theta functions, four in total, are originally expressed as an infinite series[5] in the parameter q and z. In [25] z = 0 for the second and third theta function.

The parameter j is cycled from 0 to 4 producing 5 values of S[j]. A constant value is calculated separately for S5. Let q1 = $\mathrm{Exp}[5\pi * i * \mathrm{p1}]$ and

[26] S5 = $-\mathrm{u8} * i * (\mathrm{EllipticTheta}[2,0,\mathrm{q1}]/\mathrm{EllipticTheta}[3,0,\mathrm{q1}])^{1/2}$

Let K = $-1/(2 * 5^{3/4}) * (k^2)^{1/8}/(k * (1-k^2))^{1/2}$ where K can be positive or negative (see appendix). One of the solutions of [21] is

[27] K* $(S[0] + \mathrm{S5}) * (S[1] + i * S[4]) * (i * S[2] + S[3])$

How are the remaining 4 solutions determined? The answer is permutation of S[j]! The cyclic permutation follows the rule

[28] S[0] → - i*S[3] → i*S[1] → S[4] → - S[2] → S[0] and S5 is fixed

From this permutation the next solution is

[29] K*$(-i * S[3] + \mathrm{S5}) * (-i * S[4] + i * -S[2]) * (i * -S[0] + -S[1])$

The 5 solutions are permuted to give

[30]
$$\mathrm{K}* \begin{matrix} (S[0] + \mathrm{S5}) * (S[1] + i * S[4]) * (i * S[2] + S[3]) \\ (-i * S[3] + \mathrm{S5}) * (-i * S[4] + i * -S[2]) * (i * -S[0] + -S[1]) \\ (i * S[1] + \mathrm{S5}) * (i * S[2] + i * S[0]) * (i * i * S[3] + i * S[4]) \\ (S[4] + \mathrm{S5}) * (-i * S[0] + i * -i * S[3]) * (i * -i * S[1] + -i * S[2]) \\ (-S[2] + \mathrm{S5}) * (-S[3] + i * i * S[1]) * (i * -S[4] + i * S[0]) \end{matrix}$$

Numerical solutions to [21], t=

{0.056059692478 - 1.207609145676 I},
{-0.529341417693 + 0.724299716399 I},
{1.164164579086 - 0.167227267830 I},
{-1.073471223780 - 0.272579724846 I},
{0.382588369909 + 0.923116421952 I}}

This concludes the first and second part of our strategy to obtain a solution to [3]. It is general enough to solve [3] for any constants a, b, c, d, and e. There may be some alterations in calculating S5. A good check for S5 is to compare this value with the Dedekind Eta function solution using suggested τ values. I would suggest comparing the result of [26[with the following calculation.

[31] $$\mathrm{S5} = \sqrt{2}\,\frac{\mathrm{DedekindEta}[5\mathrm{p1}/2]*(\mathrm{DedekindEta}[10*\mathrm{p1}])^2}{(\mathrm{DedekindEta}[5*\mathrm{p1}])^3}$$

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[5] Jacobi Theta Functions, Wolfram, MathWorld, Jacobi Theta Functions -- from Wolfram MathWorld

where the arguments are τ, the ratio of half periods. Adjustment of [26] may require adding or removing imaginary(i) and/or u8 terms. The eta function in [31] is available with *Mathematica*.

Next, we show a reverse strategy to transform our solutions [30] back to solutions for [3].

Transformation- retracing our steps back

The first transformation is simple, find the solution of [19]. Since z = t/f the solution is found by dividing t,[21] by f, [20].

[32] K/f*
$$\begin{matrix} (S[0]+S5)*(S[1]+i*S[4])*(i*S[2]+S[3]) \\ (-i*S[3]+S5)*(-i*S[4]+i*-S[2])*(i*-S[0]+-S[1]) \\ (i*S[1]+S5)*(i*S[2]+i*S[0])*(i*i*S[3]+i*S[4]) \\ (S[4]+S5)*(-i*S[0]+i*-i*S[3])*(i*-i*S[1]+-i*S[2]) \\ (-S[2]+S5)*(-S[3]+i*i*S[1])*(i*-S[4]+i*S[0]) \end{matrix}$$

Numerical solutions to [19], z=

$$\begin{matrix} \{-501977.61746 - 1654224.01896i\}, \\ \{-366969.826898 + 1229241.888766i\}, \\ \{1488892.19941 - 782109.64783i\}, \\ \{-1577003.58529 + 146038.09193i\}, \\ \{957058.83025 + 1061053.6861i\}\} \end{matrix}$$

Based on these complex solutions we observe that [21] and [19] could not be factored into simple terms as (z -z_1) where z_1 is an integer. If one simple root were found the equations could be factored into quartics and solved. In what follows a quartic (see [8]) does need to be solved. If we follow the trail back from [19] notice that the z in [8] refers to a solution to,

[33] $$y^4 + p*y^3 + q*y^2 + r*y + s - z = 0$$

In the first strategy of the solution, we already found the constants p, q, s, and r ([12,a,b,c] and [18b]). Since z = z1 is one of the solutions of [32] the next step is to find y in [33]. As previously demonstrated, we can depress this equation as done either manually or by program, (note here I replaced y with x).

[34] $$\text{Collect}[\text{Resultant}[x^\wedge 4 + p*x^\wedge 3 + q*x^\wedge 2 + r*x + (s - \text{z1}), y - (x^\wedge 2 + m*x + 1), x], y]$$

Where m is a constant to be determined from removing the coefficient of y^3. After substituting the 3rd solution in [32], and solving for the constant m, the new equation in y is,

[35] $$(1.1737859\times10^{14} - 3.97493\times10^{13}i) - (8.453730\times10^{10} - 2.4579454\times10^{10}i)y - (8953412.266 + 251601.247i)y^2 + y^4 = 0$$

The advantage of this equation is that it can be factored into two quadratic equations. Find s1, u1 and v1, such that

[36] $(y^2 - s1*y + u1)*(y^2 + s1*y + v1)$ = equation [35]

A suggested method is found using the Descartes method (published 1637)[6]. The method results in a resolvent cubic equation for the constant s1 where s1 = square root of s from equation [37],

[6] Solving the Quartic Equation, https://math.colorado.edu/~kearnes/Teaching/Courses/F17/quartic.pdf

[37] $$-\mathrm{qq}^2 + (\mathrm{pp}^2 - 4\mathrm{rr}) * s + 2\mathrm{pp} * s^2 + s^3 = 0$$

Here pp, qq and rr are the coefficients of y^2, y and constant, respectively in [35]. To use Cardano's method we first depress s to y in [37],

[38] $$\mathrm{Collect}[\mathrm{Resultant}[s\text{^}3 + 2\mathrm{pp} * s\text{^}2 + (-4\mathrm{rr} + \mathrm{pp}^2)s - \mathrm{qq}^2, y - (s\text{^}2 + \mathrm{mm} * s + 1), s], y]$$

Expanding and solving for mm from the coefficient of y^2 yields the following cubic.

[39] $$(-1.252784 \times 10^{44} - 1.23097342 \times 10^{45} i) + (2.973064 \times 10^{29} + 2.9447979 \times 10^{28} i) y + y^3$$

Let C1 be equal to the y coefficient and C0 the constant coefficient of [39] then using equation [15]. The cube root of the solutions are taken so as to obey the conditions [16], i.e. use the cube root of the negative of one of the roots to [15]. The square root of this solution is s1,

[40] s1 = $5947.137102192363887 - 381.346554515089110i$

u1 = $5924543.472950386838 - 789567.4295529557065i$

v1= $2.034505877808951 \times 10^7 - 3997874.3040029364i$

Once s1 is calculated v1 and u1 are obtained from linear equations. Expanding [36] confirms the coefficients in [35]. This now allows us to solve the two quadratic equations in [36] by equating to zero, resulting in four roots y_r of [35]. Each of these four roots y_r are tested by considering [34] where $y_r = x_{yr}^2 + m*x_{yr} + 1$. Returning notation of x(=y) to equation [33] we seek the confirmation that

[41] $$x_{yr}^4 + p*x_{yr}^3 + q*x_{yr}^2 + r*x_{yr} + s = z1$$

Note that s is from [12a].

Finding one significant solution of the four,

[42] $(y^2 + s1*y + v1) = 0$; $y_r = -2555.1885063404584851955 + 3613.2961034730189i$

With m = $-182.57936399652369526 68986$ from [34], solve for x,

[43] $x_{yr}^2 + m*x_{yr} + 1 = y_r$

to obtain the one solution (of eight possible solutions) of interest which is

$x_{yr} = 11.94212128446069 - 22.7687913211713109i$

and which satisfies [41]. Finally, returning to the quadratic mentioned after our quintic [3], and using m and n found in [5a,b] solve the last quadratic

[44] $$x^2 + m\,x + n - x_{yr} = 0$$

to obtain a root of [3]! Only one root of [44] satisfies [3], x = $0.4381498247200 - 1.30759868603i$ which is verified by plugging x into our example quintic $x^5 + 2x^4 + x^3 + 3x^2 + 4x - 2 = 0$.

The other roots are found by returning to equation [34] and substituting another value for z1 from the list of elliptic theta function solutions [32] and then following the path of equations [34] to [44].

Appendix

The constant K can also be expressed is another form. Using the definition of the q octic continued fraction from Chapter 46 let

[1A] $u0 = \sqrt{2}(k*(1-k^2))^{1/4}$

Then K can be re-arranged as

[2B] $K = ((\frac{k}{5^3})^{1/8}/\mathrm{u0})^2$

where K is either positive or negative.

Equation [22] for k is linked to the discriminants of the equation [2], $z^5 - z - a_0$. The argument $\frac{16}{25*\sqrt{5}*a0^2}$ represents a triangle of hypotenuse of length $25*\sqrt{5}*a_0^2$ and a side 16 units. The base of the triangle has of unit length of the square root of the discriminant of equation [2]. The hypotenuse is the unit square root of the discriminant of the equation $z^5 - a_0$. The other side of 16 units is the square root of the discriminant for the equation $z^5 + z$.

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