Q Transforms and the g Class Invariants

A continued fraction **u(τ)** studied by Ramanujan in his famous notebooks[1] is of interest and connects q-continued fractions (QCF) to modular functions and roots of certain polynomials of discriminant[2] (-d).

[1] $$u(\tau) = \sqrt{2} * q^{1/8} * \prod_{n>1} \frac{(1-q^{2n-1})}{(1-q^{4n-2})^2} = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5\ldots\ldots}}}}}}$$

Here q = $e^{2\pi i\tau}$.

When $u(\tau)$ is complex the modulus is denoted as $|u(\tau)| = u(\tau) * Conjugate\ [u(\tau)]$

Three versions of the nome, τ containing a negative integer discriminant can be used with q transforms for calculating $u(\tau)$ and the modulus of $u(\tau) = |u(\tau)|$.

[2a] $$\tau 1 = \sqrt{-d}$$

[2b] $$\tau 2 = \frac{1+\sqrt{-d}}{4}$$

[2c] $$\tau 3 = \frac{1+\sqrt{-d}}{2}$$

The q-transforms introduced in Chapter 31 are useful for converting QCFs of the modulus between the three nomes, and calculating the g class invariant for any discriminant -d and finding the k invariants associated with elliptic integrals. Four of the q-transforms are shown below. Two other transforms mentioned in Chapter 32 converting j-invariants are not discussed.

[3] $$\text{R[x]} = \sqrt{-\frac{2}{x^4} + \frac{\sqrt{2}\sqrt{2-x^4}}{x^4} + \sqrt{-1 + \frac{8}{x^8} + \frac{2}{x^4} + \frac{2\sqrt{2}}{\sqrt{2-x^4}} - \frac{8\sqrt{2}}{x^8\sqrt{2-x^4}}}}$$

[4] $$\text{CR[x]} = \left(-\frac{128(3x^8+10x^{16}+3x^{24})}{(-1+x^8)^4} + 64\sqrt{\frac{x^8+30x^{16}+255x^{24}+452x^{32}+255x^{40}+30x^{48}+x^{56}}{(-1+x^8)^8}}\right)^{1/8}$$

[5] $$\text{L2[x]} = \frac{4x^4}{(1+x^4)^2}$$

[6] $$\text{RL[x]} = \left(1 + \frac{8}{x^2} + \frac{4\sqrt{-(-2+x)^2(-1+x)}}{x^2} - \frac{8}{x}\right)^{-1/8}$$

Let q1 = $e^{2\pi i\tau 1}$ then $u(\tau 1)$ is real for all integer discriminants.

Applying the q transform CR[x] the complex $u(\tau 2)$ is converted to the real modulus $|u(\tau 2)|$:

[7] $|u(\tau 2)|$ = CR[$u(\tau 1)$]

Let $cr|u(\tau 3)|$ = CR[$u(\tau 3)$]**Conjugate*[CR[$u(\tau 3)$]] then complex $u(\tau 3)$ is converted to a real modulus:

[8] $cr|u(\tau 3)|$ = 2*CR[$u(\tau 1)$]

Applying the inverse q transform R[x] recovers the modulus $u(\tau 1)$.

$$R[cr|u(\tau 2)|/2] = R[|u(\tau 2)| = u(\tau 1)$$

From previous chapters it was shown that the g class invariant for odd discriminant -d can be calculated from $|u(\tau 2)|$. This involves obtaining a complex number $u(\tau 2)$ from [1] and converting to $|u(\tau 2)|$ with multiplication with the conjugate.

[9] $g_{-d} = (2/|u(\tau 2)|)^{1/3}$

[10] $g_{-d}^2/2 = (2/|u(\tau 3)|^2)^{1/3}$

Since $u(\tau 1)$ does not provide a g class invariant, the transforms are used to simplify the calculation of g invariants for any integer discriminant.

For even discriminant -d, the equation for g_{-d} [9], is the same as above but $|u(\tau 2)|$) is calculated from a modified q continued fraction as illustrated below using the k invariant.

The k invariant for a discriminant d is found by solving for k using the elliptic integral of the first kind discussed in Chapter 45

[11] $(\text{EllipticF}[\pi/2, 1-k^2]/(\text{EllipticF}[\pi/2, k^2]))^2 = d$

Alternatively, k is found from the QCF for $u(\tau 1)$ and the q transform L2 above directly provides a value of k.

[12] $k = L2[u(\tau 1)]^{1/2}$

This k is equivalent to the one found using the elliptic integral [11]. The advantage for this q transform is its operation works for both even and odd values of the discriminant. The k invariant is then applied to solving for $|u(\tau 2)|$.

For odd values of d:

[13] $|u(\tau 2)| = \sqrt{2} * (k * (\sqrt{1-k^2}))^{1/4}$

For even values of d:

[14] $|u(\tau 2)| = \sqrt{2} * (k/(1-k^2))^{1/4}$

The g class invariant for either even or odd d is calculated from [9] or [10]: $g_{-d} = (2/|u(\tau 2)|)^{1/3}$

For any $u(\tau 1)$ of integer discriminant d an elliptic integral relation can be found:

[15] $(\text{EllipticF}[\pi/2, 1-u(\tau 1)^8]/(\text{EllipticF}[\pi/2, u(\tau 1)^8]))^2/4 = d$

Equation [15] verifies that $u(\tau 1)$ is the correct modulus for $\tau 1$. It also provides a method for finding the real value $u(\tau 1)$ without using the QCF!

The following cascade of transforms can be used for odd d:

[16] $k_d = L2[R[CR[R[u(\tau 1)]]^{1/2}]]^{1/2}$

This k_d does not satisfy the elliptic integral equation, however $|u(\tau 2)|$ can be calculated from the relation [13] for odd values of d above. This indicates that {k, k_d } are two solutions to [13].

Combination of the above equations provides simple equations for finding any g class invariant for a given d:

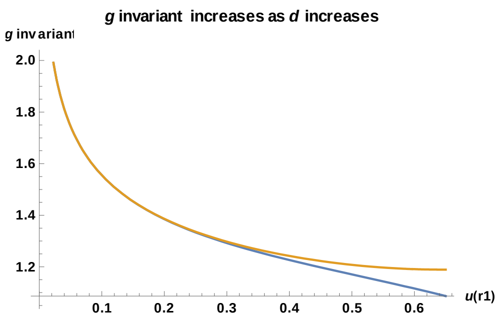
For odd values of d:

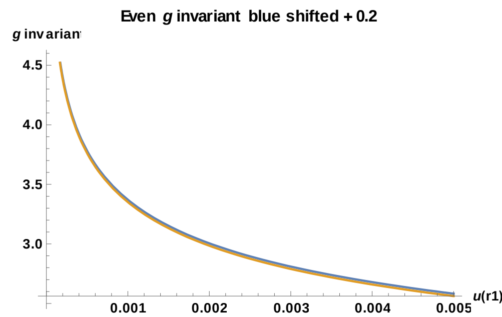
[17] $g_{-d} = 2^{1/6} * ((1 - \mathrm{L2}[u(\tau 1)])\mathrm{L2}[u(\tau 1)])^{-1/24}$

For even values of d:

[18] $g_{-d} = 2^{1/6} * (\frac{\sqrt{\mathrm{L2}[u(\tau 1)]}}{1-\mathrm{L2}[u(\tau 1)]})^{-1/12}$

The range of $u(\tau 1)$ decreases from about 0.65 for d = 1 to 0 as d increases to infinity. The two graphs below of even (blue) and odd d, illustrate how the g class invariant increases as d increases and as $u(\tau 1)$ decreases.





<u>Some inverse relations</u>

[19] $u(\tau 1)$ = **R[**$|u(\tau 2)|$**] = RL[**k^2**]**

[20] $|u(\tau 2)|$ = **CR[RL[**k^2**]]**

[21] **RL[**k_{-d}^2**] = R[CR[**$u(\tau 1)$**]]**$^{1/2}$

[22] k_{-d}^2 = **L2[R[CR[R[**$2/g_{-d}^3$**]]**$^{1/2}$**]]**$^{1/2}$

<u>Integer Sequence Structure of q transforms L2 and RL.</u>

Conversion of $u(\tau 1)$ to the k invariant using the transform L2 is associated with several integer sequences found in OEIS[3]. Let n be an integer > 1. The following sequence is obtained from $\mathrm{L2}[n]^{1/2}$ for n=1 to n = 20.

[23]

$$\{1, \frac{8}{17}, \frac{9}{41}, \frac{32}{257}, \frac{25}{313}, \frac{72}{1297}, \frac{49}{1201}, \frac{128}{4097}, \frac{81}{3281}, \frac{200}{10001}, \frac{121}{7321}, \frac{288}{20737}, \frac{169}{14281}, \frac{392}{38417}, \frac{225}{25313}, \frac{512}{65537}, \frac{289}{41761}, \frac{648}{104977}, \frac{361}{65161}, \frac{800}{160001}\}$$

The numerator is OEIS A181900 a(n)= A022998(n)*n where A022998(n) is the sequence defined if n is odd then n, otherwise 2n. For example, if n = 12 then a(12) = (2*12)*12 = 288. This sequence may explain why $|u(\tau 2)|$ obtained from [12] is dependent on whether the discriminant d is even or odd. The sequence is also closed for multiplication; (a(2)*a(3) = a(6)).

The denominator sequence is not found in OEIS, but it can be deduced that it is the largest-odd divisor of $n^4 + 1$. For example, if n = 9 then $9^4+1=6462$ and its largest- odd divisor is 3281.

The following sequence is obtained from L2[$n^{1/2}$] for n=1 to n = 20.

[24]

$$\{1,\frac{16}{25},\frac{9}{25},\frac{64}{289},\frac{25}{169},\frac{144}{1369},\frac{49}{625},\frac{256}{4225},\frac{81}{1681},\frac{400}{10201},\frac{121}{3721},\frac{576}{21025},\frac{169}{7225},\frac{784}{38809},\frac{225}{12769},\frac{1024}{66049},\frac{289}{21025},\frac{1296}{105625},\frac{361}{32761},\frac{1600}{160801}\}$$

The numerator is OEIS A154615 a(n)= $A022998(n)^2$ where A022998(n) is the sequence defined above; if n is odd then n, otherwise 2n. for example, if n = 12 then a(12) = $(2*12)^2$ = 576. This sequence may also explain why $|u(\tau 2)|^{1/2}$ is dependent on whether the discriminant d is even or odd. This sequence is also closed for multiplication; (a(3)*a(4) = a(12)).

The denominator sequence is found in OEIS A228564 when taking the square root of [24];

$$\{1,5,5,17,13,37,25,65,41,101,61,145,85,197,113,257,145,325,181,401\}$$

it is the largest-odd divisor of $n^2 + 1$. For example, if n = 9 then $9^2+1=82$ and its largest- odd divisor is 41.

The two transforms $RL[n]^{1/2}$and RL[$n^{1/2}$] both result in a series of “1’s”, in the second case requiring multiplication by the conjugate. The transform RL[1/n]**Conjugate*[RL[1/n]] results in a series of real solutions to the equations z^4- (4(n)-2)*z^2+1=0. The equation substituting $u(\tau 1)$ for n, z^4- (4($u(\tau 1)$)-2)*z^2+1=0 , when solved produces two unique real solutions z_i such that $L2[L2[z_i^{1/2}]]^{1/2}$ = k.

Other transformations with real numbers are described in Chapter 32. The q transform formula above can be verified from examples shown in Chapters 28 through 32 of **The Perrin Chalkboard**.

1. S. Ramanujan, Notebooks (2 volumes). Tata Institute of Fundamental Research, Bombay, 1957.
2. H. Weber, Table VI from **Lehrbuch der Algebra, Elliptische Funktionen und Algebraische Zahlen**, Braunschweig, Germany, 1908.
3. Online Encyclopedia of Integer Sequences

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