

Integer Sequences and Orthogonal Polynomials

Several sequence numbers can be expressed from the Sum of Cycle Indices of the Symmetry Group as described in Chapter 47 (Cycle Index of the Symmetric Group and the Jacobi Polynomial). The cycle index of symmetric group sums can be expressed using the Jacobi Polynomial or the hypergeometric function. These sequence numbers are associated with the polynomials $x^g - c_1 * x^{(g-j)} - c_0$ where g and j are integers. These sequences are termed parent (PA) and element (EL) such as the Lucas (PA) and Fibonacci (EL) sequences when $g=2$ and $j=1$. In some examples PA and EL are calculated for the polynomials $x^g - c_1 * x^{(g-j)} - c_0$ at the $(NE + 2)$ nd and $(NE + g - 2j)$ th sequence numbers, respectively, when j has no common factor of g , $\{g, j\} = 1$. A generating function for the parent and element sequences are also found in agreement with PA and EL. Note that in equations [3] and [4] below if g is even then $N1$ is also an even integer and j is not an even number. If $j > g$ then sequence numbers represent the sequence for polynomial $x^j - c_0 * x^{(j-g)} - c_1$.

Let $N1$ be an even integer. Then define NE for g even and j odd and g not divisible by j :

$$[1] \quad NE = (N2 + (j-1) * (N1 + g)) / 2$$

$$[2] \quad N2 = (N1 - g) / 2$$

Then

$$[3] \quad EL = \sum_{k=1}^{imax+1} c_1^{\frac{(j-1)*g}{j} + N2} (c_1^{-g/j} * c_0)^{k*j-1} * \text{Abs}[\text{JacobiP}[N2 - (k-1)g, g(-1+k) - jk - N2, j * k - 1, 3]]$$

$$[4] \quad PA = NE * \sum_{k=1}^{imax+1} c_1^{\frac{(j-1)*g}{j} + N2} (c_1^{-g/j} * c_0)^{k*j-1} * \text{Abs}[\text{JacobiP}[N2 - (k-1)g, g(-1+k) - jk - N2, j * k - 1, 3]] * (1 / (N2 - (k-1)(g) + j * k - 1))$$

where $imax = \text{Floor}[N2/g]$ and $\text{JacobiP}[n, a, b, 3]$ is the Jacobi Polynomial $P_n^{(a,b)}(3)$ in *Mathematica*.

Several generating functions (coefficients expanded in x^n) can be used to compare sequence numbers to EL and PA. Given $\{g, j\} = 1$ and $N1$ an even integer, the following functions are found to agree with PA and EL at the $NE + g - 2j$ and $NE + 2$ term, respectively, in *Mathematica* for equations [3] and [4].

$$[5] \quad \text{Element}(x) = \text{CoefficientList}[\text{Series}[-\frac{x^2}{-1 + c_1 * x^j + c_0 * x^g}, \{x, 0, (NE) + 2\}], x]$$

$$[6] \quad \text{Parent}(n) = \text{CoefficientList}[\text{Series}[-\frac{x^{(g-j)}(j * c_1 + c_0 * g * x^{(g-j)})}{-1 + c_1 * x^j + c_0 * x^g}, \{x, 0, NE + g - 2j\}], x]$$

Modifications to equations [3] and [4] when j is a common factor of g .

A. Let $max = \text{Floor}[(N2 + 2)/3]$, and $g/j = 2$

$$[7] \quad EL = \sum_{k=1}^{\max+4} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/2)k - N2, (j/2) * k - 1, 3]] c1^{\frac{(1-1)*g}{j} + N2} (c1^{-g/j} * c0)^{k*j/j-1} *$$

$$[8] \quad PA = (NE - g(j-1)) \sum_{k=1}^{\max+4} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/j)k - N2, (j/j) * k - 1, 3]] * c1^{\frac{(1-1)*g}{j} + N2} (c1^{-g/j} * c0)^{k*j/j-1} * (1/(N2 - (k-1)(g/j) + j/j * k - 1))$$

Equations [5] and [6] are generating functions for equations [7] and [8]. The upper limit of the sum (max + integer) may need to be adjusted when N1 increases as the sum reaches a limiting value. These equations apply to all integers j where $g = 2*j$.

B. Case j = 2, g even and equal and greater than 4.

$$[9] \quad EL = \sum_{k=1}^{\text{imax}+3} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/2, (g/2)(-1+k) - (j/2)k - N2, (j/2) * k - 1, 3]] c1^{\frac{(j/2-1)*g/2}{j/2} + N2} (c1^{-g/j} * c0)^{k*j/2-1} *$$

$$[10] \quad PA = (2N2) \sum_{k=1}^{\text{imax}+3} c1^{\frac{(j/2-1)*g/2}{j/2} + N2} (c1^{-g/j} * c0)^{k*j/2-1} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/2, (g/2)(-1+k) - (j/2)k - N2, (j/2) * k - 1, 3]] * (1/(N2 - (k-1)(g/2) + j/2 * k - 1))$$

C. Case j = 3, g even or multiple of 3. (e.g. g = 9, 12,..)

$$[11] \quad EL = \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/j)k - N2, (j/j) * k - 1, 3]] c1^{\frac{(j/3-1)*g/3}{j/3} + N2} (c1^{-g/j} * c0)^{k*j/3-1} *$$

$$[12] \quad PA = (NE - g(j-1)) \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/j)k - N2, (j/j) * k - 1, 3]] * c1^{\frac{(1-1)*g}{j} + N2} (c1^{-g/j} * c0)^{k*j/j-1} * (1/(N2 - (k-1)(g/j) + j/j * k - 1))$$

D. Case j = 4, g even and equal and greater than 6.

$$[13] \quad EL = \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/j)k - N2, (j/j) * k - 1, 3]] c1^{\frac{(j/j-1)*g/j}{j/j} + N2} (c1^{-g/j} * c0)^{k*j/j-1} *$$

$$[14] \quad PA = (NE - g(j-1)) \sum_{k=1}^{\max+3} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/j, (g/j)(-1+k) - (j/j)k - N2, (j/j) * k - 1, 3]] * c1^{\frac{(j/j-1)*g/j}{j/j} + N2} (c1^{-g/j} * c0)^{k*j/j-1} * (1/(N2 - (k-1)(g/j) + j/j * k - 1))$$

E. Case j = 6 g even > 6

$$[15] \quad EL = \sum_{k=1}^{\max+3} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/2, (g/2)(-1+k) - (j/2)k - N2, (j/2) * k - 1, 3]] c1^{\frac{(j/2-1)*g/2}{j/2} + N2} (c1^{-g/j} * c0)^{k*j/2-1} *$$

$$[16] \quad PA = (NE - \frac{g*j}{2}) * \sum_{k=1}^{\max+1} \text{Abs}[\text{JacobiP}[N2 - (k-1)g/2, (g/2)(-1+k) - (j/2)k - N2, (j/2) * k - 1, 3]] * c1^{\frac{(j/2-1)*g/2}{j/2} + N2} (c1^{-g/j} * c0)^{k*j/2-1} * (1/(N2 - (k-1)(g/2) + j/2 * k - 1))$$

For $g = 9, j = 6$ the following equations apply:

$$[15a] \text{ EL} = \sum_{k=1}^{\max+1} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/3, (g/3)(-1+k) - (j/3)k - \text{N2}, (j/3) * k - 1, 3]] c1^{\frac{(j/3-1)*g/3}{j/3} + \text{N2}} (c1^{-g/j} * c0)^{k*j/3-1} *$$

$$[16a] \text{ PA} = (\text{NE} - g(j-2)) \sum_{k=1}^{\max+1} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/3, (g/3)(-1+k) - (j/3)k - \text{N2}, (j/3) * k - 1, 3]] * c1^{\frac{(j/3-1)*g/3}{j/3} + \text{N2}} (c1^{-g/j} * c0)^{k*j/3-1} * (1/(\text{N2} - (k-1)(g/3) + j/3 * k - 1))$$

F. Case j = 8 g even > 8 g=10

$$[17] \text{ EL} = \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/2, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2) * k - 1, 3]] c1^{\frac{(j/2-1)*g/2}{j/2} + \text{N2}} (c1^{-g/j} * c0)^{k*j/2-1} *$$

$$[18] \text{ PA} = (\text{NE} - \frac{g*j}{2}) * \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/2, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2) * k - 1, 3]] * c1^{\frac{(j/2-1)*g/2}{j/2} + \text{N2}} (c1^{-g/j} * c0)^{k*j/2-1} * (1/(\text{N2} - (k-1)(g/2) + j/2 * k - 1))$$

Case j = 8, g even > 8 g=12

$$[17a] \text{ EL} = \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/4, (g/4)(-1+k) - (j/4)k - \text{N2}, (j/4) * k - 1, 3]] c1^{\frac{(j/4-1)*g/4}{j/4} + \text{N2}} (c1^{-g/j} * c0)^{k*j/4-1} *$$

$$[18a] \text{ PA} = (\text{NE} - g(j-2)) \sum_{k=1}^{\max+1} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/4, (g/4)(-1+k) - (j/4)k - \text{N2}, (j/4) * k - 1, 3]] * c1^{\frac{(j/4-1)*g/4}{j/4} + \text{N2}} (c1^{-g/j} * c0)^{k*j/4-1} * (1/(\text{N2} - (k-1)(g/4) + j/4 * k - 1))$$

G. Case j = 10 g=12

$$[19] \text{ EL} = \sum_{k=1}^{\max} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/2, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2) * k - 1, 3]] c1^{\frac{(j/2-1)*g/2}{j/2} + \text{N2}} (c1^{-g/j} * c0)^{k*j/2-1} *$$

$$[20] \text{ PA} = (\text{NE} - \frac{g*j}{2}) \sum_{k=1}^{\max+3} \text{Abs}[\text{JacobiP}[\text{N2} - (k-1)g/2, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2) * k - 1, 3]] * c1^{\frac{(j/2-1)*g/2}{j/2} + \text{N2}} (c1^{-g/j} * c0)^{k*j/2-1} * (1/(\text{N2} - (k-1)(g/2) + j/2 * k - 1))$$

Example I Calculation: g = 12, j = 10, c1 = 5, c0 = 3

Equation $x^{12} - 5*x^2 - 3$

N1 = 68

N2 = 28

NE = 388

Max = 10

EL = 1160939515680304948971703125 (equation [19])

PA = 12285208863843426899296312500 (equation [20])

The $(NE - (N1 - g + 1))^{th}$ term of equation [5] = EL

The $(NE - (N1 - g + j + 1))^{th}$ term of equation [6] = PA

As noted in the equations where g and j have a common factor (2 or 3) slight modifications in equations [3] and [4] are required. Also, the factor $(NE - \text{constant})$ that precedes PA is modified with a constant that depends on g and j . Determining sequence numbers for larger even values of $g > 12$ is simplified by using the above equations as a basis for finding expressions for $EL[g, j]$ and $PA[g, j]$ when g and j have a common factor.

Other Sequences

In all calculations above, the integer sequence also agrees with the Bell Polynomial for element and parent sequences of $x^g - c1 * x^{(g-j)} - c0$, discussed in Chapter 45. PA numbers correspond to the sum of powers of the g roots of the polynomial. However, these even polynomials in g show that the sum of powers of the g roots are spaced with zeros. For example, the polynomial $x^6 - x^3 - 1 = 0$ has a non-zero sequence value starting at $n = 3$ and has 2 intervening zeros before a non-zero value at $n = 6$. This creates an interval in the above equations between the number calculated at $N1$ and the actual location of the non-zero sequence value. In many instances the $N1$ input value agrees with the $(NE - g * (1 - j))^{th}$ value:

Example2: Let $g = 6$ and $j = 3$ ($c0 = c1 = 1$) with $N1 = 54$, equation [8] above calculates $PA = 311046$. This corresponds to the $NE + g * (1 - j) = \frac{j}{2} (N1 - g) = 72^{nd}$ value of equation [6]:

$$\text{CoefficientList}[\text{Series}[-\frac{x^{(g-j)}(j * c1 + c0 * g * x^{(g-j)})}{-1 + c1 * x^j + c0 * x^g}, \{x, 0, NE + g * (1 - j)\}], x]$$

Output = {0,0,0,3,0,0,9,0,0,12,0,0,21,0,0,33,0,0,54,0,0,87,0,0,141,0,0,228,0,0,369,0,0,597,0,0,966,0,0,1563,0,0,2529,0,0,4092,0,0,6621,0,0,10713,0,0,17334,0,0,28047,0,0,45381,0,0,73428,0,0,118809,0,0,192237,0,0,311046}

The Lucas structure of these sequences with $g/j = 2$ becomes apparent. It can also be shown that when $g/j = 3/2$ the sequences have a Perrin structure with intervening zeros.

Compare results to the powers of the six roots at $n = 72$. In *Mathematica*;

$$\text{Solve}[z^6 - z^3 - 1 == 0]$$

$$PA(72) = N[z1^{72} + z2^{72} + z3^{72} + z4^{72} + z5^{72} + z6^{72}, 6] = 311046$$

Hypergeometric formula can also be derived from the Jacobi Polynomials. An example is the hypergeometric version of equation [20] for $g = 12$ and $j = 10$ using *Mathematica*:

[21] $PA = 2 * \sum_{i=1}^{(imax+3)} \frac{(NE - \frac{g*j}{2}) / (NE - (\frac{g^2}{4}))}{c1^{\frac{(j/2-1)*g/2+N2}{j/2}} (c1^{-g/j} * c0)^{i*j/2-1} * 1 / ((N2 - (i-1)g/2) + (j/2 * i - 1)) * \text{Hypergeometric2F1}[-(N2 - (i-1)g/2), -(j/2 * i - 1), 1, 1] * (N2 + (j/2 - 1) * (N1 + g/2)/2)}$

As mentioned above, equations [3] and [4] apply to integer sequences for EL and PA when g and j are relatively prime. When g and j have a common factor [3] and [4] result in a totally different sequence not related to the parent or element sequences.

Consider the coefficient series,

$$[22] \quad \text{CoefficientList}[\text{Series}[\frac{x^4}{((1-x)^j - x^g)}, \{x, 0, (N2) + 4\}], x]$$

When $g/j = 2$ the element sequence [3] follows equation [22] with $N1$ corresponding to the $N2 + 4^{\text{th}}$ term. If $g = 6$ and $j = 3$ then equation [22] agrees with equation [3].

Example 3 $N1 = 54, N2 + 4 = 28, \text{EL}(\text{equation [3]}) = 172321$

Output (equation[22]) =

{0,0,0,0,1,3,6,10,15,21,29,42,66,111,192,330,554,906,1452,2303,3651,5826,9382,15225,24807,40431,65748,106584,172321}

The sequence does not follow a Fibonacci nor Padovan sequence but is listed in the OEIS (Online Encyclopedia of Integer Sequences) as entry A101551. Other similar sequence found there are A024490 ($g = 4, j = 2$) related to the Fibonacci cube graph and A101552 ($g = 8, j = 4$). The series can also be expressed as a binomial in g and j :

$$[23] \quad \sum_{n=1}^{\text{imax}+4} \text{Binomial}[(g-1) - j * n + N2, j * n - 1]$$

or modified when $c0$ and $c1$ are numbers other than 1,

$$[24] \quad \sum_{n=1}^{\text{imax}+4} c1^{\frac{(j-1)*g}{j} + N2} (c1^{-g/j} * c0)^{n*j-1} \text{Binomial}[(g-1) - j * n + N2, j * n - 1]$$

giving sequence numbers agreeing with equation [3] when $g/j = 2$. Dividing terms in [23] by j result in a binomial for the element numbers when $g/j = 2$

$$[25] \quad \sum_{n=1}^{\text{max}+4} \text{Binomial}[(g/j - 1) - (g/j - j/j) * n + N2, j/j * n - 1]$$

This equation is also in agreement with equation [7] above showing the correspondence of binomial coefficient with the Jacobi polynomial. (See Chapter 27 for comparison of the binomial coefficient with the Jacobi Polynomial and calculation sequence numbers for polynomials of odd powers of g .)

Jacobi and Hypergeometric Polynomials

The Jacobi function is a specialized orthogonal polynomial of the hypergeometric function. Identifying terms in $\text{JacobiP}[n, a, b, c, z]$ equation [20] shows that

$n = N2 - (k - 1)g/2, a = (g/2)(-1 + k) - (j/2)k - N2, b = (j/2) * k - 1$ and $z = 3$. The hypergeometric $2F1$ function, equation [21], is expressed as

$$[26] \quad {}_2F_1[-(N2 - (k - 1)g/2), -(j/2 * k - 1), 1, z]$$

where $z = 1$. Comparison with the polynomial JacobiP, I find;

$$[27a] \quad \text{JacobiP}[n, a, b, 3] = \text{Hypergeometric2F1}[-n, -b, 1, 1]$$

Using conditions of Example 1 above, a calculation was done using variable z on the 4th term of the hypergeometric function and on the 3rd term of the Jacobi function:

$$[27] \quad \text{JacobiP}[n, a, b * z, 3] \sim \text{Hypergeometric2F1}[-n, -b, 1, z]$$

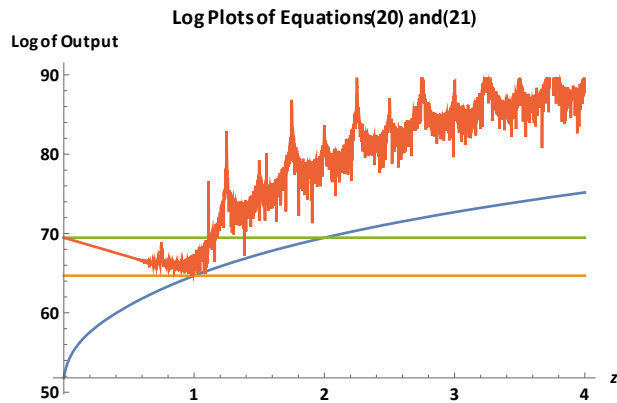


Figure 1 showing outputs of equations [20] and [21] using polynomials as a function of z (equation [27]). Blue line is hypergeometric and red line is the Jacobi. Orange and Green lines show values of equation [21] at $z = 1$ and $z = 2$, respectively.

First note that $z = 1$ both polynomials result in the same value as anticipated. At $z = 0$ I find that,

$$[28] \quad \text{JacobiP}[n, a, 0, 3] = \text{Hypergeometric2F1}[-n, -b, 1, 2]$$

The Jacobi polynomial becomes nearly chaotic at z values around 1 and above 1, however the value at $z = 1$ exactly corresponds to the hypergeometric polynomial. Conditions were then changed as shown in [29].

$$[29] \quad \text{JacobiP}[n, a, 0, 2z - 1] \sim \text{Hypergeometric2F1}[-n, -b, 1, z]$$

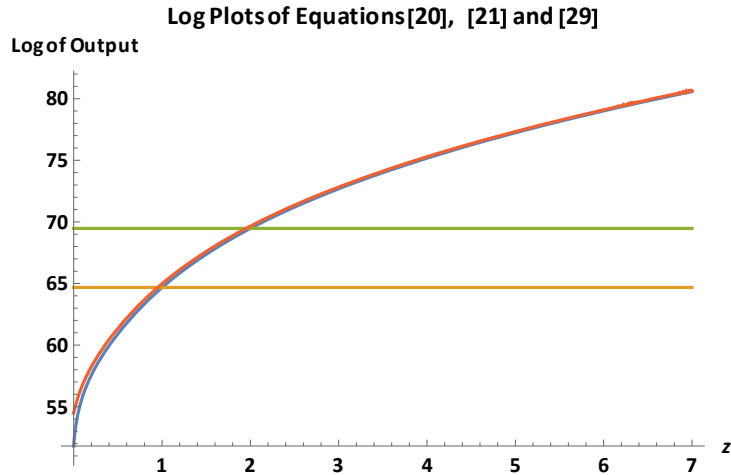


Figure 2 showing equal values for Jacobi and Hypergeometric functions when the Jacobi polynomial (red) is modified as $b = 0$ and functions with $2z - 1$. Note: Red line is shifted by a small amount (0.1 units) to show both lines.

As seen in figure 2, the chaotic fluctuations in the Jacobi Polynomial disappear when $b = 0$ for all values of z . It is also possible to substitute all Jacobi functions above with equivalent value:

$$[30] \quad \text{JacobiP}[n, a, b, 3] = \text{JacobiP}[n, a, 0, 1]$$

Then the equivalent equation for PA (Parent) as in equation [4] above becomes:

$$[4a] \quad PA = NE * \sum_{k=1}^{\text{imax}+1} c1^{\frac{(j-1)*g}{j} + N2} (c1^{-g/j} * c0)^{k*j-1} * \text{Abs}[\text{JacobiP}[N2 - (k-1)g, g(-1+k) - jk - N2, 0, 1]] * (1/(N2 - (k-1)(g) + j * k - 1))$$

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