Integer Sequences and Orthogonal Polynomials

Several sequence numbers can be expressed from the Sum of Cycle Indices of the Symmetry Group as described in Chapter 47 (Cycle Index of the Symmetric Group and the Jacobi Polynomial). The cycle index of symmetric group sums can be expressed using the Jacobi Polynomial or the hypergeometric function. These sequence numbers are associated with the polynomials x^g - c1*$x^{(g-j)}$-c0 where g and j are integers. These sequences are termed parent (PA) and element (EL) such as the Lucas (PA) and Fibonacci (EL) sequences when g =2 and j =1. In some examples PA and El are calculated for the polynomials **x^g - c1*$x^{(g-j)}$- c0** at the (NE +2)nd and (NE + g-2j) th sequence numbers, respectively, when j has no common factor of g, {g,j}=1 . A generating function for the parent and element sequences are also found in agreement with PA and EL. Note that in equations [3] and [4] below if g is even then N1 is also an even integer and j is not an even number. If j>g then sequence numbers represent the sequence for polynomial **x^j - c0*$x^{(j-g)}$- c1**.

Let N1 be an even integer. Then define NE for g even and j odd and g not divisible by j:

[1] NE = (N2+(j-1)*(N1+g)/2)

[2] N2= (N1-g)/2

Then

[3] $$\mathrm{EL} = \sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}} (\mathrm{c1}^{-g/j} * \mathrm{c0})^{k*j-1} * \mathrm{Abs}[\mathrm{JacobiP}[\mathrm{N2} - (k-1)g, g(-1+k) - jk - \mathrm{N2}, j*k-1, 3]]$$

[4] $$\mathrm{PA} = \mathrm{NE} * \sum_{k=1}^{\mathrm{imax}+1} \mathrm{c1}^{\frac{(j-1)*g}{j}+\mathrm{N2}} \left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j-1} * \mathrm{Abs}\left[\mathrm{JacobiP}[\mathrm{N2} - (k-1)g, g(-1+k) - jk - \mathrm{N2}, j*k-1, 3]\right] * (1/(\mathrm{N2} - (k-1)(g) + j*k - 1))$$

where imax = Floor[N2/g] and JacobiP[n,a,b,3] is the Jacobi Polynomial $P_n^{(a,b)}(3)$ in *Mathematica*.

Several generating functions (coefficients expanded in x^n) can be used to compare sequence numbers to EL and PA. Given {g,j} = 1 and N1 an even integer, the following functions are found to agree with PA and EL at the NE + g -2j and NE +2 term, respectively, in *Mathematica* for equations [3] and [4].

[5] $$\mathrm{Element}(x) = \mathrm{CoefficientList}[\mathrm{Series}[-\frac{x^2}{-1+\mathrm{c1}*x^j+\mathrm{c0}*x^g}, \{x, 0, (\mathrm{NE})+2\}], x]$$

[6] $$\mathrm{Parent}(n) = \mathrm{CoefficientList}[\mathrm{Series}[-\frac{x^{(g-j)}(j*\mathrm{c1}+\mathrm{c0}*g*x^{(g-j)})}{-1+\mathrm{c1}*x^j+\mathrm{c0}*x^g}, \{x, 0, \mathrm{NE}+g-2j\}], x]$$

Modifications to equations [3] and [4] when j is a common factor of g.

A. Let max=Floor[(N2+2)/3], and g/j = 2

[7] $$\mathrm{EL} = \sum_{k=1}^{\max+4} \mathrm{c1}^{\frac{(1-1)*g}{j}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/j-1} * \mathrm{Abs[JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/2)k - \mathrm{N2}, (j/2)*k - 1{,}3]]$$

[8] $$\mathrm{PA} = \left(\mathrm{NE} - g(j-1)\right) \sum_{k=1}^{\max+4} \mathrm{c1}^{\frac{(1-1)*g}{j}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/j-1} * \mathrm{Abs}\left[\mathrm{JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/j)k - \mathrm{N2}, (j/j)*k - 1{,}3]\right] * (1/(\mathrm{N2} - (k-1)(g/j) + j/j*k - 1))$$

Equations [5] and [6] are generating functions for equations [7] and [8]. The upper limit of the sum (max + integer) may need to be adjusted when N1 increases as the sum reaches a limiting value. These equations apply to all integers j where g = 2*j.

B. Case j = 2, g even and equal and greater than 4.

[9] $$\mathrm{EL} = \sum_{k=1}^{\mathrm{imax}+3} \mathrm{c1}^{\frac{(j/2-1)*g/2}{j/2}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/2-1} * \mathrm{Abs[JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \mathrm{N2}, (j/2)*k - 1{,}3]]$$

[10] $$PA = (2\mathrm{N2}) \sum_{k=1}^{\mathrm{imax}+3} \mathrm{c1}^{\frac{(j/2-1)*g/2}{j/2}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/2-1} \mathrm{Abs}\left[\mathrm{JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \mathrm{N2}, (j/2)*k - 1{,}3]\right] * (1/(\mathrm{N2} - (k-1)(g/2) + j/2*k - 1))$$

C. Case j = 3, g even or multiple of 3. (e.g. g = 9, 12,..)

[11] $$\mathrm{EL} = \sum_{k=1}^{\max} \mathrm{c1}^{\frac{(j/3-1)*g/3}{j/3}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/3-1} * \mathrm{Abs[JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/j)k - \mathrm{N2}, (j/j)*k - 1{,}3]]$$

[12] $$\mathrm{PA} = (\mathrm{NE} - g(j-1)) \sum_{k=1}^{\max} \mathrm{c1}^{\frac{(1-1)*g}{j}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/j-1} * \mathrm{Abs}\left[\mathrm{JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/j)k - \mathrm{N2}, (j/j)*k - 1{,}3]\right] * (1/(\mathrm{N2} - (k-1)(g/j) + j/j*k - 1))$$

D. Case j = 4, g even and equal and greater than 6.

[13] $$\mathrm{EL} = \sum_{k=1}^{\max} \mathrm{c1}^{\frac{(j/j-1)*g/j}{j/j}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/j-1} * \mathrm{Abs[JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/j)k - \mathrm{N2}, (j/j)*k - 1{,}3]]$$

[14] $$\mathrm{PA} = (\mathrm{NE} - g(j-1)) \sum_{k=1}^{\max+3} \mathrm{c1}^{\frac{(j/j-1)*g/j}{j/j}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/j-1} * \mathrm{Abs}\left[\mathrm{JacobiP[N2} - (k-1)\,g/j\,, (g/j)(-1+k) - (j/j)k - \mathrm{N2}, (j/j)*k - 1{,}3]\right] * (1/(\mathrm{N2} - (k-1)(g/j) + j/j*k - 1))$$

E. Case j = 6 g even > 6

[15] $$\mathrm{EL} = \sum_{k=1}^{\max+3} \mathrm{c1}^{\frac{(j/2-1)*g/2}{j/2}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/2-1} * \mathrm{Abs[JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \mathrm{N2}, (j/2)*k - 1{,}3]]$$

[16] $$\mathrm{PA} = (\mathrm{NE} - \tfrac{g*j}{2}) * \sum_{k=1}^{\max+1} \mathrm{c1}^{\frac{(j/2-1)*g/2}{j/2}+\mathrm{N2}}\left(\mathrm{c1}^{-g/j} * \mathrm{c0}\right)^{k*j/2-1} * \mathrm{Abs}\left[\mathrm{JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \mathrm{N2}, (j/2)*k - 1{,}3]\right] * (1/(\mathrm{N2} - (k-1)(g/2) + j/2*k - 1))$$

For g = 9, j = 6 the following equations apply:

[15a] $\text{EL} = \sum_{k=1}^{\max+1} \text{c1}^{\frac{(j/3-1)*g/3}{j/3}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/3-1} * \text{Abs[JacobiP[N2} - (k-1)\,g/3\,, (g/3)(-1+k) - (j/3)k - \text{N2}, (j/3)*k - 1{,}3]]$

[16a] $\text{PA} = (\text{NE} - g(j-2)) \sum_{k=1}^{\max+1} \text{c1}^{\frac{(j/3-1)*g/3}{j/3}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/3-1} * \text{Abs}\left[\text{JacobiP[N2} - (k-1)\,g/3\,, (g/3)(-1+k) - (j/3)k - \text{N2}, (j/3)*k - 1{,}3]\right] * (1/(\text{N2} - (k-1)(g/3) + j/3*k - 1))$

F. Case j = 8 g even > 8 g=10

[17] $\text{EL} = \sum_{k=1}^{\max} \text{c1}^{\frac{(j/2-1)*g/2}{j/2}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/2-1} * \text{Abs[JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2)*k - 1{,}3]]$

[18] $\text{PA} = (\text{NE} - \frac{g*j}{2}) * \sum_{k=1}^{\max} \text{c1}^{\frac{(j/2-1)*g/2}{j/2}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/2-1} * \text{Abs}\left[\text{JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2)*k - 1{,}3]\right] * (1/(\text{N2} - (k-1)(g/2) + j/2*k - 1))$

Case j = 8, g even > 8 g=12

[17a] $\text{EL} = \sum_{k=1}^{\max} \text{c1}^{\frac{(j/4-1)*g/4}{j/4}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/4-1} * \text{Abs[JacobiP[N2} - (k-1)\,g/4\,, (g/4)(-1+k) - (j/4)k - \text{N2}, (j/4)*k - 1{,}3]]$

[18a] $\text{PA} = (\text{NE} - g(j-2)) \sum_{k=1}^{\max+1} \text{c1}^{\frac{(j/4-1)*g/4}{j/4}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/4-1} * \text{Abs}\left[\text{JacobiP[N2} - (k-1)\,g/4\,, (g/4)(-1+k) - (j/4)k - \text{N2}, (j/4)*k - 1{,}3]\right] * (1/(\text{N2} - (k-1)(g/4) + j/4*k - 1))$

G. Case j = 10 g=12

[19] $\text{EL} = \sum_{k=1}^{\max} \text{c1}^{\frac{(j/2-1)*g/2}{j/2}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/2-1} * \text{Abs[JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2)*k - 1{,}3]]$

[20] $\text{PA} = (\text{NE} - \frac{g*j}{2}) \sum_{k=1}^{\text{imax}+3} \text{c1}^{\frac{(j/2-1)*g/2}{j/2}+\text{N2}}\left(\text{c1}^{-g/j}*\text{c0}\right)^{k*j/2-1} * \text{Abs}\left[\text{JacobiP[N2} - (k-1)\,g/2\,, (g/2)(-1+k) - (j/2)k - \text{N2}, (j/2)*k - 1{,}3]\right] * (1/(\text{N2} - (k-1)(g/2) + j/2*k - 1))$

Example I Calculation: g = 12, j = 10, c1 = 5, c0 = 3

Equation $x^{12} - 5*x^2 - 3$

N1 = 68

N2 = 28

NE = 388

Max = 10

EL = 1160939515680304948971703125 (equation [19])

PA = 12285208863843426899296312500 (equation [20])

The (NE –(N1-g+1))th term of equation [5] =EL

The (NE –(N1-g+j+1))th term of equation [6] = PA

As noted in the equations where g and j have a common factor (2 or 3) slight modifications in equations [3] and [4] are required. Also, the factor (NE – constant) that precedes PA is modified with a constant that depends on g and j. Determining sequence numbers for larger even values of g >12 is simplified by using the above equations as a basis for finding expressions for EL[g,j] and PA[g,j] when g and j have a common factor.

Other Sequences

In all calculations above, the integer sequence also agrees with the Bell Polynomial for element and parent sequences of x^g - c1*$x^{(g-j)}$- c0, discussed in Chapter 45. PA numbers correspond to the sum of powers of the g roots of the polynomial. However, these even polynomials in g show that the sum of powers of the g roots are spaced with zeros. For example, the polynomial x^6-x^3-1=0 has a non-zero sequence value starting at n = 3 and has 2 intervening zeros before a non-zero value at n = 6. This creates an interval in the above equations between the number calculated at N1 and the actual location of the non-zero sequence value. In many instances the N1 input value agrees with the (NE -g*(1-j))th value:

Example2: Let g = 6 and j = 3 (c0 = c1 = 1) with N1 = 54, equation [8] above calculates PA = 311046. This corresponds to the $\mathrm{NE} + g * (1 - j) = \frac{j}{2}(\mathrm{N1} - g)$ = 72nd value of equation [6]:

$$\mathrm{CoefficientList}[\mathrm{Series}[-\frac{x^{(g-j)}(j * \mathrm{c1} + \mathrm{c0} * g * x^{(g-j)})}{-1 + \mathrm{c1} * x^j + \mathrm{c0} * x^g}, \{x, 0, \mathrm{NE} + g * (1 - j)\}], x]$$

Output = {0,0,0,3,0,0,9,0,0,12,0,0,21,0,0,33,0,0,54,0,0,87,0,0,141,0,0,228,0,0,369,0,0,597,0,0,966,0,0,

1563,0,0,2529,0,0,4092,0,0,6621,0,0,10713,0,0,17334,0,0,28047,0,0,45381,0,0,73428,0,0,118809,0,0,192237,0,0,311046}

The Lucas structure of these sequences with g/j = 2 becomes apparent. It can also be shown that when g/j = 3/2 the sequences have a Perrin structure with intervening zeros.

Compare results to the powers of the six roots at n = 72. In *Mathematica*;

$$\mathrm{Solve}[z^6 - z^3 - 1 == 0]$$

$$\mathrm{PA}(72) = N[\mathrm{z1}^{72} + \mathrm{z2}^{72} + \mathrm{z3}^{72} + \mathrm{z4}^{72} + \mathrm{z5}^{72} + \mathrm{z6}^{72}, 6] = 311046$$

Hypergeometric formula can also be derived from the Jacobi Polynomials. An example is the hypergeometric version of equation [20] for g = 12 and j = 10 using *Mathematica*:

[21] $$\mathrm{PA} = 2 * (\mathrm{NE} - \tfrac{g*j}{2})/(\mathrm{NE} - (\tfrac{g^2}{4})) \sum_{i=1}^{(\mathrm{imax}+3)} \mathrm{c1}^{\frac{(j/2-1)*g/2}{j/2}+\mathrm{N2}} (\mathrm{c1}^{-g/j} * \mathrm{c0})^{i*j/2-1} * 1/((\mathrm{N2} - (i-1)\,g/2) + (j/2 * i - 1)) * \mathrm{Hypergeometric2F1}[-(\mathrm{N2} - (i-1)\,g/2), -(j/2 * i - 1), 1, 1] * (\mathrm{N2} + (j/2 - 1) * (\mathrm{N1} + g/2)/2)$$

As mentioned above, equations [3] and [4] apply to integer sequences for EL and PA when g and j are relatively prime. When g and j have a common factor [3] and [4] result in a totally different sequence not related to the parent or element sequences.

Consider the coefficient series,

[22] $$\text{CoefficientList}[\text{Series}[\frac{x^4}{((1-x)^j-x^g)},\{x,0,(\text{N2})+4\}],x]$$

When g/j = 2 the element sequence [3] follows equation [22] with N1 corresponding to the N2 +4th term. If g = 6 and j = 3 then equation [22] agrees with equation [3].

Example 3 N1 = 54, N2 +4 = 28, EL(equation [3]) = 172321

Output (equation[22]) =
{0,0,0,0,1,3,6,10,15,21,29,42,66,111,192,330,554,906,1452,2303,3651,5826,9382,15225,24807,40431,65748,106584,172321}

The sequence does not follow a Fibonacci nor Padovan sequence but is listed in the OEIS (Online Encyclopedia of Integer Sequences) as entry A101551. Other similar sequence found there are A024490 (g = 4, j = 2) related to the Fibonacci cube graph and A101552 (g = 8, j = 4). The series can also be expressed as a binomial in g and j:

[23] $$\sum_{n=1}^{\text{imax}+4}\text{Binomial}[(g-1)-j*n+\text{N2},j*n-1]$$

or modified when c0 and c1 are numbers other than 1,

[24] $$\sum_{n=1}^{\text{imax}+4}\text{c1}^{\frac{(j-1)*g}{j}+\text{N2}}(\text{c1}^{-g/j}*\text{c0})^{n*j-1}\text{Binomial}[(g-1)-j*n+\text{N2},j*n-1]$$

giving sequence numbers agreeing with equation [3] when g/j = 2. Dividing terms in [23] by j result in a binomial for the element numbers when g/j = 2

[25] $$\sum_{n=1}^{\text{max}+4}\text{Binomial}[(g/j-1)-(g/j-j/j)*n+\text{N2},j/j*n-1]$$

This equation is also in agreement with equation [7[above showing the correspondence of binomial coefficient with the Jacobi polynomial. (See Chapter 27 for comparison of the binomial coefficient with the Jacobi Polynomial and calculation sequence numbers for polynomials of odd powers of g.)

Jacobi and Hypergeometric Polynomials

The jacobi function is a specialized orthogonal polynomial of the hypergeometric function. Identifying terms in JacobiP[n,a,b,c,z] equation [20] shows that

n = $\text{N2}-(k-1)\,g/2$, a = $(g/2)(-1+k)-(j/2)k-\text{N2}$, b = $(j/2)*k-1$ and z = 3. The hypergeometric2F1 function, equation [21], is expressed as

[26] $${}_2\mathbf{F}_1[[-(\text{N2}-(k-1)\,g/2),-(j/2*k-1),1,z]$$

where z = 1. Comparison with the polynomial JacobiP, I find;

[27a] $\text{JacobiP}[\text{n}, a, b, 3] = \text{Hypergeometric2F1}[-n, -b, 1,1]$

Using conditions of Example 1 above, a calculation was done using variable z on the 4th term of the hypergeometric function and on the 3rd term of the Jacobi function:

[27] $\text{JacobiP}[\text{n}, a, b * z, 3] \sim \text{Hypergeometric2F1}[-n, -b, 1, z]$

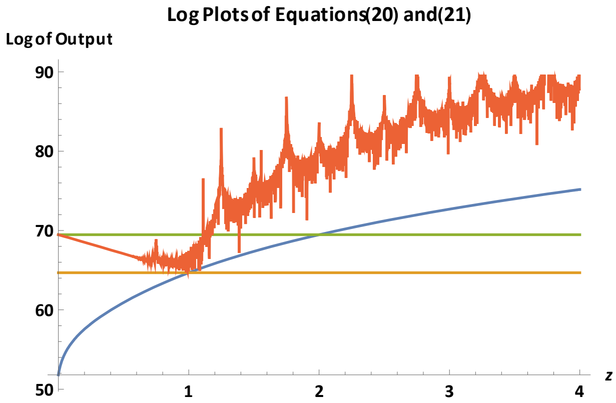

Figure 1 showing outputs of equations [20] and [21] using polynomials as a function of z (equation [27]). Blue line is hypergeometric and red line is the Jacobi. Orange and Green lines show values of equation [21] at z = 1 and z =2, respectively.

First note that z =1 both polynomials result in the same value as anticipated. At z = 0 I find that,

[28] $\text{JacobiP}[\text{n}, a, 0,3] = \text{Hypergeometric2F1}[-n, -b, 1,2]$

The Jacobi polynomial becomes nearly chaotic at z values around 1 and above 1, however the value at z =1 exactly corresponds to the hypergeometric polynomial. Conditions were then changed as shown in [29].

[29] $\text{JacobiP}[\text{n}, a, 0, 2z - 1] \sim \text{Hypergeometric2F1}[-n, -b, 1, z]$

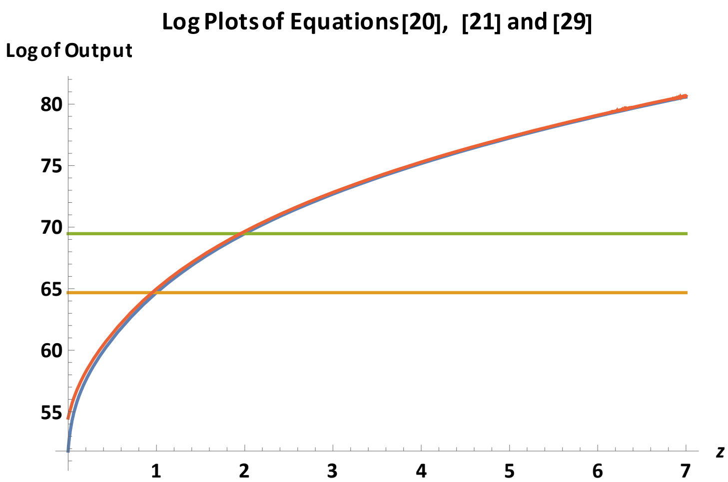

Figure 2 showing equal values for Jacobi and Hypergeometric functions when the Jacobi polynomial (red) is modified as b = 0 and functions with 2z -1. Note: Red line is shifted by a small amount (0.1 units) to show both lines.

As seen in figure 2, the chaotic fluctuations in the Jacobi Polynomial disappear when b = 0 for all values of z. It is also possible to substitute all Jacobi functions above with equivalent value:

[30] $\text{JacobiP}[\text{n}, a, b, 3] \ = \ \text{JacobiP}[\text{n}, a, 0,1]$

Then the equivalent equation for PA (Parent) as in equation [4] above becomes:

[4a] $\text{PA} = \text{NE} * \sum_{k=1}^{\text{imax}+1} \text{c1}^{\frac{(j-1)*g}{j}+\text{N2}} \left(\text{c1}^{-g/j} * \text{c0}\right)^{k*j-1} * \text{Abs}\left[\text{JacobiP}[\text{N2} - (k-1)g, g(-1+k) - jk - \text{N2}, 0,1]\right] * (1/(\text{N2} - (k-1)(g) + j * k - 1))$

R. Turk November 18, 2022

Special Cases

Equation [8] above can be expanded into two parts by substituting the equations for NE (eqn. [1]) and N2 (eqn. [2]) in terms of the input N1. Where N1 can be either an even or odd integer.

[31] F = $(\text{g1}\,(-2 + \text{j1})/2) * \sum_{k=1}^{\text{imax1}+1} \text{c11}^{\frac{(\text{j1}-1)*\text{g1}}{\text{j1}}+\text{N21}} (\text{c11}^{-\text{g1}/\text{j1}} * \text{c01})^{k*\text{j1}-1} * \text{Abs}[\text{JacobiP}[\text{N21} - (k-1)\text{g1}, \text{g1}(-1+k) - \text{j1}k - \text{N21}, 0,1] *$

$(1/(\text{N21} - (k-1)(\text{g1}) + \text{j1} * k - 1)) + \text{N1}\,(\text{j1})/2 * \sum_{k=1}^{\text{imax1}+1} \text{c11}^{\frac{(\text{j1}-1)*\text{g1}}{\text{j1}}+\text{N21}} (\text{c11}^{-\text{g1}/\text{j1}} * \text{c01})^{k*\text{j1}-1} *$

$$\text{Abs}[\text{JacobiP}[\text{N21} - (k-1)\text{g1}, \text{g1}(-1+k) - \text{j1}k - \text{N21}, 0, 1]] * (1/(\text{N21} - (k-1)(\text{g1}) + \text{j1} * k - 1))$$

Represented as **F = F1 + N1 * F2**

When j1 = 2 we find F1 = 0 then F = N1* F2. This is true of the Perrin sequence g= 3, j = 2 indicating F2 is the sigma orbit described in previous chapters.

Let x_i be solutions to the equation $x^g - c1*x^{(g-j)} - c0 = 0$. Then by definition, sequence numbers for this equation are given as;

[32] $$\sum_{k=1}^{g} \text{x(k)}^n$$

Where n is a positive integer. Negative sequences are also defined when n is a negative integer.

Given a sequence generated from g and j and c1 and c0, equation [31] for F calculates sequence numbers in agreement with [32] for integers n > 0. The negative sequence numbers, in agreement with [32] for integers n< 0, are found from equation [31] when substituting gn = g , jn = |g-j| and c1 = -c1/c0 and c0 = 1/c0 where gn and jn are positive numbers.

For example, the positive sequence numbers for the equation $x^4 - 3*x^{(4-1)} - 5 = x^4 - 3*x^{(3)} - 5 = 0$ is g = 4, j = 1 c1 = 3 and c0 = 5. Then, to find the negative sequence numbers from [31] let gn= 4, jn =4-1 = 3 and c1=-3/5 and c0=1/5. The positive and negative sequence numbers are calculated by eqn. [32] from the roots of the original positive equation $x^4 - 3*x^{(3)} - 5 = 0$. As shown in previous chapters the sequence numbers for negative sequences can be either integers or fractional rational numbers but either is calculated from the Jacobi Polynomial expression [31].

Since j determines the separation of sequence numbers expressed by equation [31], a match in the input N1 and N1n where N1 is not equal to N1n, is required for these numbers to agree with eqn. [32] for a given n and -n. The value of NE is useful to find these integers. Once NE is calculated from N1, g and j, it is equated to the expression for NEn using gn and jn to find a value of N1n. The value must be an integer so Tables of NE and NEn can be used to find when NE = NEn for a given value of N1. Then, n = NE and -n = -NE values will agree with equation [31].

A curious result occurs when j<0 and g>0. Consider g = 4 and j =-3. Then this represents the equation for g1 = g +|j| = 7 and j1 = g1+j = 4 or $x^7 - c1*x^{(7-4)} - c0 = 0$. The negative sequence is calculated from gn = 7, jn = gn - jn = 3 and c1 = -c1/c0 and c0 = 1/c0 as above.

A sequence generated from g and j< 0 with |j|< g is equivalent to a sequence with g1 = g +|j| and j1 = g1+j representing the integer sequence numbers for roots of $x^{g1} - c1*x^{(g1-j1)} - c0 = 0$.

An example is found in OEIS A050443 which is a fourth order sequence related to the Perrin sequence. It is generated from g = 3 and j = -1 with c1 = c0 =1.

A special case occurs when j = -g. Then g1 = g +|-g| = 2g and j1 = g1+j = 2g – g = g or g1/ j1 = 2.

A sequence generated from j = -g is equivalent to a sequence with g1 = 2g and g1/j1 = 2. The Jacobi Polynomial used to represent this sequence is obtained from an appropriate modification of equation [32] based on cases A thru G above.

The Binomial Transform

The last special case of sequence numbers which equation [31] calculates is related to the binomial transform of a sequence. When g = j, both positive numbers, various new sequences are obtained which are not like the sequences described by $x^g - x^{(g-j)} - 1 = 0$. When g = j = 2 the sequence F(n)/2 is $\{1,2,4,8,16,32,64,128,..\}$, the sequence of numbers 2^n. When g= 3 the sequence F(n)/g is $\{1,3,6,11,21,42,85..\}$ not a power series. In OEIS A024495 this sequence is given various formula and comments regarding its generation. When g = 4 the sequence F(n)/g is $\{0,0,0,1,4,10,20,36,64,120 \ldots\}$ and found in OEIS A00749. A comment in this entry indicates it is the binomial transform of the period 4 repeat: [0,0,0,1]. Although this is correct the comment should include the statement that it is the first difference in the binomial transform.

In order to generate the period 4; [0,0,0,1], take the sequence of numbers modulus (4); [1,2,3,0,1,2,3,0..]. *Mathematica* uses the Boole function to convert true statements into 1s and false statements into 0s. Then the command Boole[Mod[n,4]] creates b1[n] = $\{0,0,0,1,0,0,0,1,0,0,0,1 \ldots.\}$ as needed. The binomial has been previously used in *Mathematica* as Binomial[n,k], then,

[33] $\text{Table}[\sum_{k=0}^{n} \text{Binomial}[n,k] * \text{b1}[k], \{n, 0,14\}] = \{1,1,1,1,2,6,16,36,72,136,256,496,992,2016,4096\}$

produces the first 15 numbers in the binomial transform of period 4. Taking first differences in this sequence gives $\{0,0,0,1,4,10,20,36,64,120,240,496,1024,2080\}$, which is the sequence obtained from eqn. [31] for F(n)/4. Note: differences are obtained by appending" //Differences" to eqn. [33]

For any integer g when g = j, b1[n] = Boole[Mod[n,g]] and equation [33] and F[n]/g are both in agreement. As an extreme test given g = j = 33 and n = 43, eqn. [31] calculates F[43]/33 = 435897 and [33] shows 32 zeros preceding the sequence $\{1,33,561,6545,58905,435897\}$. It can be shown that F[n]/g always agrees with the $(N2+g)^{th}$ term in eqn. [33].

The Binomial Transform offers us an opportunity to look for other sequences which are transformed from a binary input. In our discussion of Perrin pseudoprimes in previous chapters, I discovered a binary sequence which has a period of 14. It would be interesting to find the resulting sequence of its transform. Given the procedure of Boole operations on the modulus of a number it is possible to break the sequence into its various parts. The sequence produces 1s at 6, 10, 12, and 14 and repeats every 14 units. Using *Mathematica,* the sequence is reproduced as,

[34] rt[n] := Boole[Mod[n + 14, 14] == 6] + Boole[Mod[n + 14, 14] == 10] + Boole[Mod[n + 14, 14] == 12] + Boole[Mod[n + 14, 14] == 0]

Rt[n] = $\{[0,0,0,0,0,1,0,0,0,1,0,1,0,1,0,0,0,0,0,1,0,0,0,1,0,1]\}$

Taking the first difference of the sequence gives,

[35] $a[n] = \text{Table}[\sum_{k=0}^{n} \text{Binomial}[n,k] * \text{b1}[k], \{n, 0{,}26\}]$= {0, 0, 0, 0, 1, 6, 21, 56, 127, 262, 518, 1024, 2081, 4382, 9478, 20736, 45254, 97580, 206721, 428964, 870695, 1728146, 3355839, 6384824, 11931206}

This sequence was not found in OEIS. However, taking the numbers to modulus (2) gives a reverse binary sequence of period 14,

[36] Mod[a [n],2] = $\{[0,0,0,0,1,0,1,0,1,0,0,0,1,0,0,0,0,0,1,0,1,0,1,0,0,0,1]\}$

The limiting ratio of terms a [n+1]/ a [n] in the sequence of eqn. [35] as n increases is 2.

Also, the binary transform is found to be a linear combination of each term in the function rt[n]. And each term is the sum of binomials of (m-1)+14 where m is 6, 10, 12 or 0. For example, the binomial transform of the first term in [34], rt6[n] = Boole[Mod[n + 14, 14] == 6] is given by the sum of binomials,

[37] $\text{Table}[\text{Binomial}[n, 5] + \text{Binomial}[n, 19] + \text{Binomial}[n, 33], \ldots + \text{Binomial}[n, 5 + 14 * k]$

It is noteworthy that the Jacobi Polynomial (and hypergeometric function) in [31] is directly related to a binomial term when g = j. The binomial function was first introduced with the Jacobi polynomial. It has been shown that the Perrin sequence and associated sequences are simple sequences that have revealed a deeper mathematical structure than originally expected. There is still more research to be done in the study of integer sequences and their relation to hypergeometric and modular functions which I have introduced in this book.

R. Turk 1/29/2023