The Complex Octagonal Geometry of Sequences

There is a paradox in describing sequences. In Chapter 20, I discussed the solution of various cubic equations using quadratics and the Dedekind function for certain discriminants. The number or order of the quadratic field (Z($\sqrt{-D}$)) can be viewed as the discriminant of a quadratic equation with solutions of the quadratic form $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where b^2-4ac = -D for certain combinations of [a,b,c]. The solutions of the irreducible cubic polynomials are obtained from the Dedekind Eta function,

[1] $$f3(-D) = |\,\mathrm{DedekindEta}[\tau 2]/\mathrm{DedekindEta}[\tau 1]| * \frac{\sqrt{2a1}}{\sqrt{2a2}}$$

With the complex numbers defined from the quadratic field

[2] $$\tau 1 = \frac{b1+\sqrt{b1^2-4a1c1}}{2a1} \text{ and } \tau 2 = \frac{b2+\sqrt{b2^2-4a2c2}}{2a2}$$

As an example, I find for D = -211, {[a,b,c]} = {[1,1,53] [5,3,11]} and f3(-211) = x^2 + 1 where x is the real root of the irreducible polynomial x^3 -3x^2 +x -2. The definition of “discriminant” for this cubic is calculated from one real and two imaginary roots xa, xb, and xc,

[3] $$((\mathrm{xa}-\mathrm{xb}) * (\mathrm{xa}-\mathrm{xc}) * (\mathrm{xb}-\mathrm{xc}))^2 = -211$$

Although the real root x can be expressed by f3(-211) as shown in the Tables of Chapter 20, only 4 discriminants {-23,-31,-59, and -83} show that f3(-D) = x based on the quadratic formula.

I demonstrate in later chapters how the Weber function is simplified by using a new choice of the quadratic field independent of knowing the quadratic forms above. If a complex number τ3 is defined,

[4] $$\tau 3 = \frac{1+\sqrt{-D}}{4}$$

the Dedekind Eta function calculates a new complex number R1 = r1+ r2*i,

[5] R1 = $\sqrt{2} * (\mathrm{DedekindEta}[t3] * (\mathrm{DedekindEta}[4 * t3])^2)/(\mathrm{DedekindEta}[2 * t3])^3$

This complex number when multiplied by its complex conjugate R1* = r1- r2*i, results in a real number which has been shown to be an algebraic integer which is converted to the solution of the Weber g type by a simple equation.

[6] $$g(-D) = (2/(R1R1^*))^{1/3}$$

Although D = -211 in both calculations, g(-D) does not equal x above and has no simple connection (more about this below). If there are two real but unconnected solutions using the same discriminant D= -211, then how can we bring these discriminants into agreement? Do they represent the same invariant to polynomial equations?

I demonstrated in previous chapters that the Ramanujan q-octic continued fraction (see Chapter 28, Eqn[2]) provides the identical complex number as R1 in equation[5] using the same complex number $\tau 3$. I use the symbol CR2 for the real product of R1 with its complex conjugate R1*. From equation [6] both g(-D) and CR2 = R1R1* are interchangeable.

If we exchange variables of g(-D) for f3(-D) in equation[1] it is possible to obtain a CR2(f3) value for the solution f3(-211).

[7] $\mathrm{CR2(f3)} = 2/(\mathrm{f3}(-D))^3$

In Chapters 31 and 46 on Elliptic functions I demonstrate methods to convert CR2 values to the modulus k of the elliptic integral and converting the elliptic integral to the equivalent discriminant. The "equivalent discriminant" or De conforms to the discriminant used to calculate the q-octic continued fraction or as found by equation[3] above.

For the conversion of any real solution to a polynomial in x it a simple calculation to start from x or f3(-D) and find a new -D=-De that conforms to the polynomial in x and to the solution as obtained from the q-octic continued fraction. Given CR2(f3), k and k' = $(1-k^2)^{1/2}$ are solved from either equation [2A] or [3A],Chapter 46 (dependent on whether an even or odd discriminant is used):

[8, 3A] $$\mathrm{CR2(f3)} = \sqrt{2}(k * k')^{1/4}$$

The value of k is then plugged into equation[1A], Chapter 46:

[9, 1A] $$\left(\frac{\mathrm{EllipticF}[\pi/2, 1-k^2]}{\mathrm{EllipticF}[\pi/2, k^2]}\right)^2 = De$$

As seen in equations[8] and [9] the discriminant De as defined, differs from the definition in equation[3]. It should be noted that I have previously shown the q transform described in Chapter 49 compares CR2 values of $\tau3$ and $\tau1$ above but is not as general as equations[7] to [9] for converting any real number to an equivalent discriminant. There is also consideration of determining the j-invariant.

Note that De =D in equation[3] only if CR2(f3) = CR2. For class order 3 cubic polynomials this was found only for discriminants -23,-31,-59, and -83. The study of the Perrin sequence associated with D = -23 was essential to finding this equivalence.

If we look at the example of D = -211 it can be shown that the equivalent discriminant to f3(-211) is an irrational number De= -292.21358687599.. When De is substituted into the q-octic fraction, CR2(f3) is recovered and x can be calculated from $(2/(\mathrm{CR2(f3)})^{1/3} = x^2 + 1$ in agreement with the real solution to the irreducible polynomial $x^3 - 3x^2 + x - 2 = 0$.

One unanswered question remains, is there a relation between the real solutions g(-De) and x(D)? The answer is yes and no. For D=-211, it is possible to show that g(-De) is the real solution to the cubic equation $(-2) - (2\mathrm{x})z - (2\mathrm{x})z^2 + z^3 = 0$ where x is the real root xa in equation[3]. For other values of D the form of the cubic can lead to a relation but may require coefficients of z and z^2 that are irrational numbers expressed as radicals in $D^{(1/2)}$. In the cubic class of order 3 the minimal polynomials of g(-De) are mainly 9th order equations. A cubic equation in z of the above form can be found by calculating the product (z-g(-De)*(z-zb)*(z-zb*) where zb and zb* are a set of complex and conjugate solutions of the 9th order equation. Choosing the correct conjugate pair out of 4 possible pairs results in a cubic of form $-2-Az-Bz^2-z^3$.It is found that A/xa and B/xa may be integer, rational or irrational numbers. When D = -211, I find A/xa = B/xa = 2.

We next look at the properties of the complex number R1. What are the criteria for R1 to be a suitable q-octic fraction for an equivalent discriminant De? Although R1 is a complex number it has special properties to identify itself as a complex number associated with the discriminant De. Again, Elliptic functions are closely associated with R1= r1+ r2i. First, a complex number is a two- dimensional construct which is a number found on the complex plane. The real axis and imaginary axis define the point of a complex number on this plane (r1,r2). An imaginary number can also be expressed in exponential form as a modulus multiplied by an argument. The modulus is $(R1R1^*)^{1/2}$ or squaring the modulus gives the number that is labeled CR2. The argument provides the phase angle ϕ in radians of R1 on the complex plane and is a number between +/- π. In exponential form R1 = $\mathrm{CR2}^{1/2} * e^{i\emptyset}$.

The octal properties of R1 are best illustrated by two equations of identity. The first is an identity for the eighth power of R1.

[10] $(R1^8 - 1) * \mathrm{Conjugate}[R1^8 - 1] = 1$

The right-hand side is exactly the integer "1" and a q-octic complex number is required to satisfy this equation. The left side is composed of two numbers which have significance as I will be illustrating below.

The second equation identifying R1 is its octal property on the elliptic function.

[11] $(\frac{\mathrm{EllipticF}[\pi/2, 1-R1^8]}{\mathrm{EllipticF}[\pi/2, R1^8]})^2 * 4 = (|De| - 1) - 2 * |De|^{1/2}$

where |De| is the absolute value of a negative number. The discriminant De does not need to be an integer but can be any real negative number satisfying the octic continued fraction. For example, both De = -211 or De= $-292.21358687599..$ are appropriate discriminants satisfying the algebraic integers g{-211) and f3(-211). The precision of this equation is dependent on the precision of R1 which can be computed to over 100 decimal places with *Mathematica* (see Notes*).

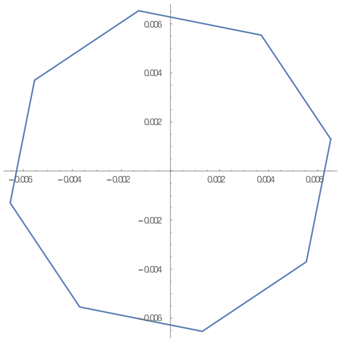
The octal property of the q-octic continued fraction can also be found in the number of equivalent solutions to equations [10] and [11] when the complex number R1 is rotated by 45 degrees on the complex plane. A rotation operator on R1 maintains the modulus but rotates the phase angle by 45 degrees. Using R1 at De = -211 as an example apply, the operator $(-1)^{1/4}$ once to obtain,

[12] R1 = $0.0816292..e^{i0.19634954..}$ and $(-1)^{1/4}$ R1 =$0.0816292..e^{i0.98174770..}$

showing that the modulus remains constant as the phase angle increases by 45 degrees. Convert the phase angles to degrees by multiplying each by 180/ π . Note that R1 has a phase angle close to 11.25 degrees and the operator changes this angle to close to 56.25 degrees but the difference is exactly 45 degrees. As |De| increases R1 approaches 11.25 degrees and the rotation is always 45 degrees. It can be shown that the rotation operator on R1 does not change the conditions found in equations [10] and [11]. If we continue to apply the rotation operator 8 times, we come back to the original value of R1. Note that $8*45^o = 360^o$ which is a full rotation. This can be shown using a polar plot of the points on the graph of $[\emptyset, CR2]$.

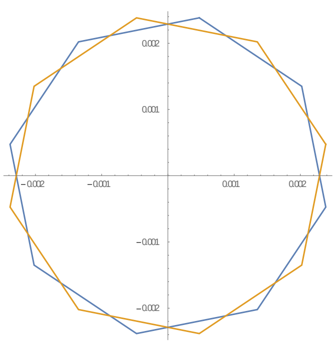
A polar plot illustrates the complex octagonal geometry of the q-octic continued fraction. Note the initial rotation of the octagon by 11.25 degrees from vertical.

Polar Plot of the rotations of R1 by 45 degrees. Each vertex point satisfies the identity tests of equations [10] and [11]. The radius of the octagon to each vertex is CR2 = R1R1*



Let R2 be the complex q-octic continued fraction determined from De= −292.21358687599.. Then the octagonal properties of R2 are similar to those of R1 except the radius to the vertices is reduced at larger discriminants: CR2(f3) <CR2. For both R1 and R2 it is possible to find a second independent octagon which also satisfies the identity criteria. A second set of vertices can be found at a midpoint phase angle shifted by 45 + 22.5 = 67.5 degrees which is a midpoint between two vertices of the first set. This second set is completely independent of the first set. The new rotation operator is found as the conjugate of -R1 or -R2 times the imaginary number i, or Conjugate[-R1i]. Once this rotation operator is performed then the original 45-degree rotation operator is applied 8 times to complete the cycle. A comparison of the two sets of rotation is shown on the following polar plot.

Polar Plot of the rotations of R2 by 45 degrees (orange) and by 67.5 degrees followed by 45 degrees (blue). Each vertex point satisfies the identity tests of equations [10] and [11]. The radius of the octagon to each vertex is CR2{f3) from equation[7].



The second set of vertices can also be produced by transposition of the real value with the complex value of the first set; r1+r2i becomes r2+r1i. This indicates there is a symmetry in the 16 values which comprise a solution to the q-octic continued fraction.

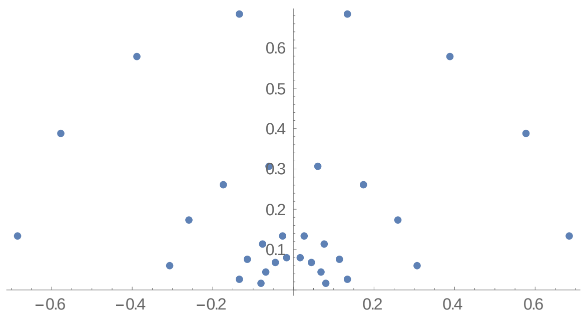
Some of this symmetry is found in equation[10]. If the 8 vertices of each set are added the result is 0. If they are multiplied together and 1 is subtracted, then one set equals the result ($R1^8$ - 1) and the other set equals the Conjugate[$R1^8$ – 1].

[13] ($R1^8$ - 1) = $\left(-\prod_{j=0}^{7}(-1)^{j/4} * \mathrm{R1} - 1\right)$ and Conjugate[$R1^8$ – 1] = $(-\prod_{j=0}^{7}(-1)^{j/4} * \mathrm{R1b} - 1)1$

where R1b = Conjugate[-R1i].

If all the points on the upper half of the complex plane are plotted as a polar plot, we see that they form concentric circles with the outer circles of larger radius for lower discriminants and the inner circles of reduced radius at increasing discriminants.

Polar plot of octagonal vertices forming half circles in the upper half of the complex plane of the two sets of q-octic continued fractions at four discriminant values. Increasing diameters are shown for De =-211, -141, -59 and -13.



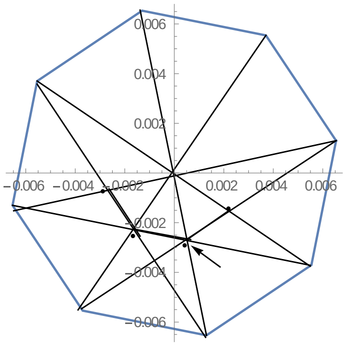
There is a slight upward bend in the radial lines from (0,0) out to the lower discriminants. The circles are assumed to reach a limit of 0 at high |De| and a limiting radius at De = -1 where CR2 = $2^{(1/4)}$. Note that g(-1) = CR2(-1) based on eqn. [6]. This view of the octagons in the complex plane suggests that if plotted in a 3^{rd} dimension to include discriminant, we would visualize the upper surface of a tunnel entering at a radius of $2^{(1/4)}$ and vanishing at (0,0) at infinite |De|. Within the selected vertices are R1 = $\mathrm{CR2}^{1/2} * e^{i\emptyset}$ values for an infinite number of discriminants representing real algebraic integer solutions expressed by,

[14] g(-D) = $(2/(\mathrm{R1}/e^{\mathrm{i}\emptyset})^2)^{1/3}$

at 8 different vertices found at (R1, Ø) of the octagons in the upper complex plane. As shown for D = -211 duplicate polynomials of equal discriminant are provided a distinct value of an equivalent discriminant De, and the R1 values lie on a separate octagon of radius CR2. Each octagon then is represented by a discriminant and an algebraic integer g as in eqn. [14]. An algebraic integer is a real solution to a polynomial equation and each octagon represents a potential polynomial of some integer order, z^n. In some instances, the polynomial may be known, and a real root is located by calculation. Using equations [8] and [9] allow calculation of the discriminant. Equation [11] or the q-octic can then determine R1 and its rotated values.

Within the octagon we can find an inner octagon by drawing chords to each vertex. In the figure below a section of the inner octagon can be seen and new vertices indicated by 4 dots and with an arrow.

The creation of an inner octagon within the octagon of radius CR2 and vertices rotated from R1



The geometry of the octagon has been well studied since ancient times. If we are given the radius of the side of a wedge given by CR2, then the length of an edge a0 is,

[15] $$\mathrm{a0} = \mathrm{CR2} * 1/(\sqrt{4+2\sqrt{2}}/2)$$

If we want to calculate the radius of the inner octagon, we see that it is the distance along the radial line to the vertex indicated by an arrow. The chord extending from one vertex to the opposite vertex splits the octagon in half and bisects the outer angle which is known to be 135 degrees. At the arrow we notice this chord also bisects the inner octagon so the angle at the arrow is a bisection of 135 degrees resulting in an isosceles triangle with two sides of length a0 and the third angle = 180- 2*135/2 = 45 degrees. Obtain the distance from the arrow to the outer vertex using triangle geometry to find the distance is (2-$\sqrt{2}$)*CR2 so the radius of the inner octagon is CR2-(2-$\sqrt{2}$)*CR2 = (-1 + $\sqrt{2}$)*CR2. The ratio of the inner octagon to the outer octagon radius is then (-1 + $\sqrt{2}$). (Hint- *Mathematica* calculates coordinates of a triangle knowing 2 angles and 1 edge, AASTriangle[3 Pi/8, 3 Pi/12, a0] with angles in radians.

This ratio allows us to calculate g(-De) values for the inner radius knowing the outer radius. It also can work in reverse assuming an octagon of given radius is the inner octagon of a larger octagon. As an example, consider an outer octagon for De = -7. The q-octic continued fraction value for CR2 = $\frac{1}{\sqrt{2}}$ with g(-7) = $\sqrt{2}$.

The inner octagon CR2io = $\frac{1}{\sqrt{2}} * (-1 + \sqrt{2})$. Using equations [8] and [9] we get De = $-23.9319085514835671..$ The q-octic continued fraction confirms the value for CR2io = $0.29289321881..$ = $\frac{1}{2}(2-\sqrt{2})$ = $\frac{1}{\sqrt{2}} * (-1 + \sqrt{2})$. Based on this new CR2 value for the inner octagon g(io) = $2^{2/3}(2-\sqrt{2})^{-1/3}$. Then g(io) = g(-7)* $(1+\sqrt{2})^{1/3}$. This equation is true for calculating the algebraic integer associated with the inner octagon for any g(De)! Finding the outer octagon g(oo) for g(-7) , g(oo) = g(-7)/ $(1+\sqrt{2})^{1/3}$ = $(2(2-\sqrt{2}))^{1/3}$. Although this value may be correct it cannot be verified by the q octic since the associated CR2oo value exceeds the value for De = -1 where CR2oo>$2^{1/4}$ and equations [8] and [9] do not work properly when |De|<1 or CR2>$2^{1/4}$.

However, there is some light shining on this problem. If we search for the minimal polynomial for g(io) and g(oo) we find it is the same polynomial! This is a 6th order polynomial: $8 - 8z^3 + z^6 = 0$ where both g(io) and g(oo) are real roots! The remaining 4 roots are complex so would not be found by this method. Also, to resolve the q-octic problem take the inverse of CR2oo and obtain a number less than $2^{1/4}$. A new De is calculated from equations [8] and [9], De=-9.77517442.. Then once R1 is calculated let CR2oo = 1 /(R1R1*). This result using equation[6] results in the expected g(oo)= $(2(2-\sqrt{2}))^{1/3}$. Based on this observation, the tunnel model previously mentioned has a symmetry for radii CR2 and 1/CR2 when CR2>$2^{1/4}$. |De| increases again once CR2>$2^{1/4}$.

The discriminant for the minimal polynomial $8-8z^3+z^6$ is D = 1528823808. If we cube g(oi) and g(oo) then these numbers are shown to be roots of a quadratic equation $8-8z+z^2$ = 0 with a discriminant D = 32. De is always negative since the q-octic continued fraction calculates complex infinities for positive De. This example illustrates that minimal polynomials with either positive or negative discriminant D can be converted to negative De.

Another interesting feature of the inner and outer octagons is the relation between g(De), g(io) and g(oo) as defined above for the real root of the minimal polynomial:

[16] $g(De)^2 = g(io)*g(oo)$

This equation applies to any root that is surrounded by roots of an upper and lower octagon. As expected, this also applies to the corresponding CR2 values:

[17] $CR2(De)^2 = CR2(io)*CR2(oo)$

It also applies to the complex values of R1

[18] $R1(DE)^2 = R1(io)*R1(oo)$

The geometry of the 3 octagons interconnects the values of g, CR2 and R1. It also illustrates the existence of inner and outer discriminants De_i and De_o connected to an integer discriminant De. Although inner and outer discriminants can be reproduced by any multiple or divisor of g(De) and give similar results as equations [16] to [18], a geometrical proof of such objects supports the hypothesis of an infinity of connected octagons on the complex plane. Currently there exists no simple calculation on CR2 or g that brings inner and outer *integer* discriminants together: g(De-1), g(De) and g(De+1). Although the Ramanujan ladder that I describe in previous chapters connects g with CR2 (g^3*CR2 = 2) for any discriminant, the proportionality between 2 discriminants requires knowledge of 3 variables to find the 4th variable.

Why are there 8 equivalent points on two sets of octagons (see Note*)? From a geometrical perspective the points on the octagon on a complex plane could also be considered to originate as an octagon from a slice on a 3-dimensional octahedron or rotated octahedron. A rotation of 90 or 180 degrees in the (x, y) plane is indistinguishable from the same rotation into the z plane if the octagon sits in the middle of an octahedron. From a physical model of the octagon there are symmetries equivalent to a dihedral D_8 group such as used for modeling the chemical compound cyclooctene. It is also possible to model the octahedron as an O_8 group used for the modeling of chemical complexes around a central charged metal. If one compares the mathematical to the physical model, then R1 represents eigenstates and CR2 is the square of the modulus or eigenvalue of these states. This idea is speculation and would require a contrast of the complex values calculated from the q-octic continued fraction with quantum calculations that agree with experiment when vectors are made complex.

**Note on Rotation of the complex values from the q octic continued fraction.*

*The q-octic continued fraction, R1, for a given discriminant is calculated in **Mathematica** from equations [1A] to [3A];*

[1A] $t2 = \frac{1}{4}(1 + i\sqrt{D})$

[2A] $q2 = Exp[2 * Pi * i * (t2)]$

[3A] $R1 = \sqrt{2} * (q2)^{1/8}/(1 + ContinuedFractionK[(q2)^n, 1 + (q2)^n, \{n, 1{,}5000\}])$

Let R1= a+bi from equation[3A]. The factor $\sqrt{2}$ in this equation provides the unrotated value (a,b). The following factors replace $\sqrt{2}$ to provide the 7 rotations: (1+i), i$\sqrt{2}$, (i-1), -$\sqrt{2}$, (-1-i), -i$\sqrt{2}$, (1-i). The first rotation takes R1 to the second solution R2 = c+di, (c,d). The remaining rotations are (-b,a), (-d,c), (-a,-b), (-c,-d), (b,-a) and (d,-c). A 67.5-degree rotation provides the root of a second set of vertices. Use equation [4A] in place of [3A].

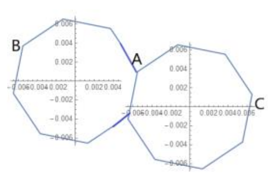
[4A] $Conjugate[[-i(\sqrt{2}) * (q2)^{1/8}/(1 + ContinuedFractionK[(q2)^n, 1 + q2^n, \{n, 1{,}5000\}])$

Equation [4A] calculates b+ai, (b,a). The following factors replace ($\sqrt{2}$) to provide the 7 rotations for the second set of solutions: (1-i), -i$\sqrt{2}$, (-i-1), -$\sqrt{2}$, (-1+i), i$\sqrt{2}$, (1+i). These remaining rotations are (-c,d), (-a,b), (-d,-c), (-b,-a), (c,-d), (a,-b),and (d,c).

R. Turk 9-11-2023

Appendix – Proof that Octagons of different De are concentric and not stacked.

Consider a manifold in which the octagons are stacked and complex values R1 can be found on attached octagons as in the following figure.



Let A be a complex point of the left octagon Ra = r1a+r2a*i. Point B is obtained by the operation Rb = $(-1)^{(3/4)}$*Ra = r3b +r4b*i. Point C can sit on the complex plane with coordinates obtained from Ra and Rb. The real distance between A and C is r1a +|r3b|. The imaginary distance is r4b*i -r2a*i. The coordinates of point C from the origin of the left octagon can be found as Rc = (r1a+ (r1a +|r3b|),r2a*i – (r4b*i -r2a*i)).

Because the angle between the left side origin and point C is not an angle of +/-11.25 *n degrees it does not satisfy equation [10]. However, the modulus squared,CR2c = Rc* Conjugate[Rc] can be calculated and a real value g(De) is calculated. It can be shown that equations [8] and [9] can be used to rotate the value of Rc into the same coordinate system as Ra and Rb. The new discriminant De' obtained from eqn[9] is used in the q-octic continued fraction to obtain the rotated value Rc'.

Using several discriminants, I find that CR2c = Rc* Conjugate[Rc] = Rc'* Conjugate[Rc']. Also comparing values to the original octagon where CR2A = Ra* Conjugate[Ra] the following relations between CR2c and CR2a and ga(De) and gc(De') are found:

[1a] CR2c/CR2a = $5 + 2\sqrt{2}$

[2a] gc(De') = g(De)a/ $(5 + 2\sqrt{2})^{1/3}$

The ratio or Rc'/Rc shows that the modulus of both complex numbers is the same, and Rc is rotated by about 14.6 degrees to obtain Rc'. It should be noted that Rc' obeys equations [8] and [9] and from [1a] the new octagon with Rc' is concentric to the original octagon with Ra.

Connections to Hypergeometric and Modular functions

In Chapter 50 I discuss the observation that the cycle index of symmetric group sums can be expressed using either the Jacobi Polynomial or the hypergeometric function. The Jacobi function is a specialized orthogonal polynomial of the hypergeometric function. I find also that the hypergeometric ${}_2F_1[a, b: c; z]$ and the Jacobi polynomial $P_n^{a,b}(x)$ can express the complete Elliptic integral of the 1st kind used in equations [9] and [11] above[1].

[3a] $\mathrm{EllipticF}[\pi/2\,, z^2] = \pi/2 * \mathrm{Hypergeometric2F1}[1/2\,, 1/2\,, 1, z^2]$

[4a] $\mathrm{Hypergeometric2F1}[1/2\,, 1/2\,, 1, z^2] = \mathrm{JacobiP}[-1/2\,, 0{,}0, -(2z^2 - 1)]$

The appropriate ratio of elliptic integrals can be replaced with hypergeometric or Jacobi polynomials of the form above. It is interesting how these functions are closely related to algebraic integers and to solutions of polynomials of positive and negative discriminants.

If we calculate the ratio of hypergeometric functions as in [5a] we obtain a tau (τ) value with an interesting property. Tau is a ratio of half periods of a doubly periodic modular function.

[5a] $\tau = i * \mathrm{Hypergeometric2F1}[1/2\,, 1/2\,, 1{,}1 - \mathrm{R1}]/(\mathrm{Hypergeometric2F1}[1/2\,, 1/2\,, 1, \mathrm{R1}]$

Here R1 is the value obtained from the q octic continued fraction. Back in Chapter 28 I showed how the ratio of Dedekind Eta functions was used to find R1.

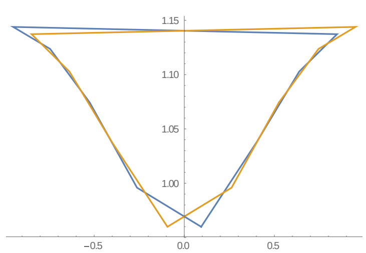
[6a] $\mathrm{R1} = \sqrt{2}\mathrm{DedekindEta}[\mathrm{t2}] * (\mathrm{DedekindEta}[4 * \mathrm{t2}])^2/(\mathrm{DedekindEta}[2 * \mathrm{t2}])^3$

In this case, tau is equal to $\mathrm{t2} = \frac{1}{4}(1 + i\sqrt{De})$. It can also be shown that the tau value from [5a] can reproduce R1 from equation [7a];

[7a] $\mathrm{R1} = 16/((\mathrm{DedekindEta}[\tau/2]/\mathrm{DedekindEta}[2\tau])^8 + 16)$

Note that tau and t2 are not equal. Tau is a modulus from the complete Elliptic integral of the first kind that is equivalent to equation[5a]. It is also the modulus used in the argument of the modular lambda function described below. The value of τ changes as the value of R1 is rotated. It is not clear which modular function τ represents (Weierstrass elliptic function?) and why tau changes with rotation of R1.

Plot showing τ as R1 is rotated. Axies are the real (x) and imaginary (y) components of τ. (Rotations of R1 (blue) and rotations of Conjugate[-R1*i] (orange)). All values lie on the positive complex plane.



[1] See H. Bateman, Higher Transcendental Functions, Vol II, McGraw Hill, 1953, p170.

The changing values of tau when applied to the modular Lambda always return the correct rotated R1 value.

[8a] R1 = ModularLambda$[\tau]$

where *Mathematica* terminology is used for the calculations. (Note: the ratio of Theta functions can also be used to represent the modular lambda function, however in *Mathematica* the nome, q = $\mathrm{Exp}[\pi * i * \tau]$ is used as the argument). This nome is also used in the q-expansion of the modular lambda function in powers of q.

A variation in the modular lambda called the *star* Lambda (Λ*) surprisingly results in a simple equation to directly obtain the Weber g class invariants for both even and odd discriminants.

[9a] Λ^* (De) = ModularLambda$[i * De]$

As an example, find G_{23} and its root.

[10a] Λ^* (23) = $(\mathrm{ModularLambda}[i * \sqrt{23}])^{1/2}$

= 0.00213993178964..

The *star* Lamda calculates the modulus of the Elliptic function found in equation [9] above. The formula for Weber's f3(De) in [7] are, G_{De} = $\mathrm{Sin}[2 * \mathrm{ArcSin}[\Lambda^*(\mathrm{De})]]^{-1/12}$ and for g_{De} = $\mathrm{Tan}[2 * \mathrm{ArcTan}[\Lambda^*(\mathrm{De})]]^{-1/12}$ where G_{De} is used for odd discriminants and g_{De} for even discriminants. The trigonometric identity is used to express equations [8] and [7] avoiding calculation of complex R1.

The Weber g class invariant is directly obtained from the star Lambda by multiplying by $2^{(1/4)}$.

[11a] $\mathrm{Sin}[2 * \mathrm{ArcSin}[\Lambda^*(23)]]^{-1/12}$ = 1.57536402 ...

[11b] $2^{(1/4)} * \mathrm{Sin}[2 * \mathrm{ArcSin}[\Lambda^*(23)]]^{-1/12}$ = 1.8734341..

As previously shown, dividing by $\sqrt{2}$, results in the plastic number, a solution to $z^3 - z - 1 = 0$. Star values are always lower, and the corresponding G and g values are obtained by multiplication with $2^{(1/4)}$.

The discriminant is shown to be the dependent variable in equations [9a] to [11a] above. The modular lambda function takes the place of the q-octic continued fraction circumventing the complex elliptic modulus R1 and indirectly calculates the G, g invariants directly, bypassing the modulus squared calculation,CR2 = R1R1*. The radius of the octagon would not have been suggested if we had started our analysis of polynomial equations using the modular lambda function. Also, equation [11] finds the discriminant through the elliptic modulus (or hypergeometric ratio) and implies the structure of the invariant algebraic integers as a set of complex octagons. Remember that each algebraic integer obtained from a q octic continued fraction is the limiting value of the ratio of the $n^{th}/(n-1)^{th}$ sequence number. I hope these new insights into linear recurrences are incentives for further research in this area of mathematics.

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