Chapter 54 The Law of Cosines and Generation of Sequences from Phasors

In the previous Chapter I discussed polynomials, sequences, and their relation to phasors. The Perrin sequence is used as an example, where the Perrin polynomial $x^3 - x - 1$ has 3 roots, one real and two complex conjugates. Each of these roots can be described as equations on a complex ellipse.

The equations for these complex and real ellipses are expressed by the magnitude times an exponential of the imaginary argument,

[1] $(-1)^{j/4} * \mathrm{Abs}[\mathrm{Rc}] * \mathrm{Exp}[i * \phi]$

[2] $(-1)^{j/4} * \mathrm{Abs}[\mathrm{Rr}] * \mathrm{Exp}[i * Arg[Rr]]$

The number $(-1)^{j/4}$ is applied to rotate the vertices on the octagon in the complex plane with j ranging from 0 to 8. The plot of these vertices produces two octagons at different radius and tilts the octagon about the (x,y) plane. All vertices are contained in a complex ellipse with a magnitude given by the term Abs[Rc] and Abs[Rr] (*Mathematica*).

The complete phasor expression for these polynomials and the sequence number m are given by:

[3] $\mathrm{Abs}[2(\mathrm{Rr}^{-2}\mathrm{Exp}[2i\mathrm{Arg}[\mathrm{Rr}]])^{1/3}]^m + ((-1)^{j1/4} * \mathrm{Abs}[\mathrm{Rc}] * \mathrm{Exp}[i * (\phi)])^m + ((-1)^{j2/4} * \mathrm{Abs}[\mathrm{Rc}] * \mathrm{Exp}[i * (-\phi)])^m$

The terms j1 and j2 are integers from 0 to 8 such that j1 + j2 = 8 and properly rotate the octagons for the roots of the polynomial.

I will demonstrate that summation of the terms of a phasor can be expressed with a trigonometric function which can simplify equation [3]. If we want to sum both complex terms which are conjugates there exists a trigonometric function that is equivalent.

Consider the terms for the Perrin sequence when m = 1,

[4] $((-1)^{3/4} * \mathrm{Abs}[\mathrm{Rc}] * \mathrm{Exp}[i * (\phi)])^1 + ((-1)^{5/4} * \mathrm{Abs}[\mathrm{Rc}] * \mathrm{Exp}[i * (-\phi)])^1$

Define the leading terms as A1 and A1, then [4] is

[5] $(A1 * \mathrm{Exp}[i * (\phi)])^1 + (\mathrm{A2} * \mathrm{Exp}[i * (-\phi)])^1$

For the complex roots Rc, this expression is the negative of the plastic ratio or the real solution to Perrin's polynomial above. Using the law of cosines on the complex plane the summation in [5] should agree with.

[6] $((\mathrm{A1}^2 + \mathrm{A2}^2 + 2\mathrm{A1} * \mathrm{A2} * \mathrm{Cos}[2\phi]))^{1/2}$

However, the complex values of A1 and A2 in [6] make this equation false. By taking the absolute values (magnitudes) of Abs[A1] and Abs[A2] and rotating the argument by 90 degrees [7] agrees with [5].

[7] $-((\mathrm{Abs}[A1]^2 + \mathrm{Abs}[A2]^2 + 2\mathrm{Abs}[A1] * \mathrm{Abs}[A2] * \mathrm{Cos}[-\mathrm{Pi}/2 + 2\phi]))^{1/2}$

Since $\mathrm{Cos}[-\mathrm{Pi}/2 + 2\phi]$ = $\mathrm{Sin}[2\phi]$ equation [7] is equal to A3(1)

[8] A3(1) = $-((\mathrm{Abs}[A1]^2 + \mathrm{Abs}[A2]^2 + 2\mathrm{Abs}[A1] * \mathrm{Abs}[A2] * \mathrm{Sin}[2\phi]))^{1/2}$

A3(m) is a real number and when added to the real root of the Perrin polynomial calculates the value of the sequence for m = 1 which is zero. Since A3 is a real number, it is important that the magnitudes of A1 and A2 are used in equation [8].

When this above method is applied to higher sequence numbers (e.g. k = 2) I find that these magnitudes must be raised to the power k.

[9] A3{m} = (sgn) $((A1^k)^2 + (A2^k)^2(\mathrm{sgn})\ 2A1^2 * A1^2 * \mathrm{Sin}[2k\phi] or \mathrm{Cos}[2k\phi]))^{1/2}$

Note that the sgn (+ or minus) and the function (Sin or Cos) will be dependent on the power k. For Perrin's sequence Sin is used for odd k and Cos for even k.

The real part of [3] can also be streamlined as

[10] $\mathrm{Abs}[2(\mathrm{Rr}^{-2}\mathrm{Exp}[2i\mathrm{Arg}[\mathrm{Rr}]])^{1/3}]^m$ = $\mathrm{Abs}[(2\mathrm{Rr}^{-2})^{k/3}]$

since the Abs value of the exponential argument is equal to one.

There is one further simplification to equation [9],

[11] A3(k odd) = $sgn(sgna * 2\mathrm{Abs}[\mathrm{Rc}]^{2k}(sgna\ 1 + \mathrm{Sin}[2k\phi]))^{1/2}$

[12] A3(k even) = $sgn(sgna * 2\mathrm{Abs}[\mathrm{Rc}]^{2k}(sgna\ 1 + \mathrm{Cos}[2k\phi]))^{1/2}$

where the sgn and sgna is plus or minus.

The complete Perrin sequence is expressed as numbers in k.

[13] $\mathrm{Abs}[(2\mathrm{Rr}^{-2})^{k/3}]$ +A3(k)

For k = 17

[14a] $\mathrm{Abs}[(2\mathrm{Rr}^{-2})^{17/3}] - (2\mathrm{Abs}[\mathrm{Rc}]^{34}(1 + \mathrm{Sin}[34\phi]))^{1/2}$

For k = 18

[14b] $\mathrm{Abs}[(2\mathrm{Rr}^{-2})^{18/3}] + (-2\mathrm{Abs}[\mathrm{Rc}]^{36}(-1 + \mathrm{Cos}[36\phi]))^{1/2}$

The sgn and sgna follow a particular cycle depending on the sequence for various discriminants.

As required for the power series using roots of a polynomial the values of Rr and Rc must be determined. Let xr be the real and xc and xcc be the conjugate complex roots.

[15a] $Rr = (2/xr^3)^{1/2}$

[15b] $Rc = (xc*xcc)^{1/2}$

The equations[11] and [12] require an argument ϕ. Chapter 52 provides a complicated method using the value of Rc and solving for a modulus, using the modulus in an elliptic function to find an equivalent discriminant, and then using the Jacobi function to obtain a value of Rc on the ellipse.

There are 2 methods which can provide the value for the argument.

1. Rotation of the argument of xc. Use Arg[$(-1)^{j/4}$ *xc] in *Mathematica* and vary j from 0 to 8. The minimum value is ϕ.
2. Rearrange equation [11] or [12] and solve for the angle ϕ.

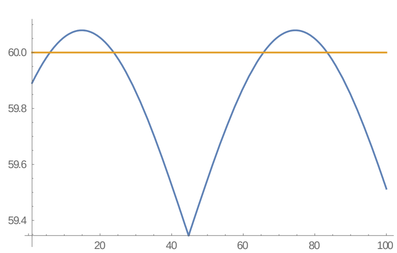
   [16] $\phi = \mathrm{ArcCos}[(\frac{(\mathrm{Pk}-\frac{2^{k/3}}{\mathrm{Abs}[\mathrm{Rr}]^{2k/3}})^2}{-2\mathrm{Abs}[\mathrm{Rc}]^{2k}}+1)]\Big/(2k)$

   One value of a non-zero sequence number for integer k must be known. In the case of the Perrin sequence the ArcCos is used for even k and ArcSin for odd k. Some guesswork is involved with this method.

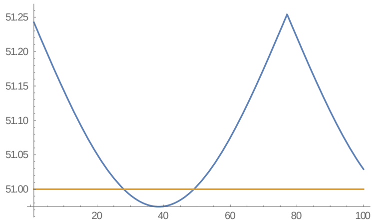
Several cycle plots of the Cosine function of argument times n for D = -17, D = -23,

D = -31and D = -59 with sequence number (line) shown below.

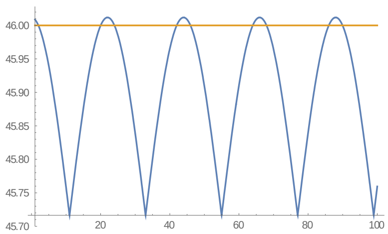
Plot of $\frac{2^{3/3}}{\mathrm{Abs}[\mathrm{R17}]^{6/3}} + (2\mathrm{Abs}[\mathrm{FR17}]^6(1+\mathrm{Sin}[n(\phi 17+\mathrm{Pi})]))^{1/2}$



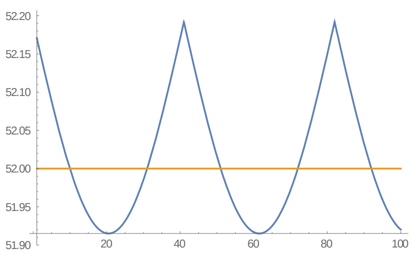
Plot of $\frac{2^{14/3}}{\text{Abs[R23]}^{28/3}} - (-2\text{Abs[FR23]}^{28}(-1 + \text{Cos}[n * \phi 23]))^{1/2}$



Plot of $\frac{2^{10/3}}{\text{Abs[R31]}^{20/3}} + (2\text{Abs[FR31]}^{20}(1 + \text{Cos}[n * \phi 31]))^{1/2}$



Plot of $\frac{2^{5/3}}{\text{Abs[R59]}^{10/3}} - (-2\text{Abs[FR59]}^{10}(-1 + \text{Cos}[n * \phi 59]))^{1/2}$



Note that for a given n value there are harmonics at lower and higher arguments that provide an integer value. These harmonics are not necessarily integers that are integer multiples of n. Integer sequences are a summation of real number vectors (roots) which can increase in magnitude to integers as they are multiplied together and summed. The progression may be viewed as a sting of numbers that consistently give integer values:

a+b+c, aa+(bb+cc), aaa+(bbb+ccc), aaaa+(bbbb+cccc), ...

where for a Perrin sequence a = xr, b =xc and c = xcc. The eigenvalues (roots) of a given polynomial originate in complex ellipses that have magnitudes Rr and Rc, where Arg[Rc] must be at an angle to the complex plane described by equations [11], [12] and [16].For each power k this angle shifts from 180 degrees to 0 degrees for the complex solutions such that the total angle aligns like a magnet along the real axis directing the kth number of the sequence to a positive or negative integer.